The Membrane Model of Black Holes and Applications

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Abstract

In my lectures I shall give an introduction to the "membrane model" of black holes (BHs). This is not a new theory, but a useful reformulation of the standard relativistic theory of BH –as far as physics outside the horizon is concerned– which is much closer to the intuition we have gained from other fields of physics. While the basic equations look less elegant, it has the advantage that we can understand astrophysical processes near a BH much more easily. In the membrane model one first splits the elegant 4-dimensional physical laws into space and time (3+1 splitting). For a stationary BH there is a preferred decomposition. Relative to this the dynamical variables (electromagnetic fields, etc.) become quantities of an "absolute space" which evolve as functions of an "absolute time", as we are accustomed to from nonrelativistics physics. We shall see, for example, that the 3+1 splitting brings Maxwell's equations into a form which resembles the familiar form of Maxwell's equations for moving conductors. We can then use the pictures and the experience from ordinary electrodynamics.

In a second step, one replaces the boundary conditions at the horizon by physical properties (electric conductivity, etc.) of a fictitious membrane. This procedure is completely adequate as long as one is not interested in fine details very close to the horizon. The details of this boundary layer are, however, completely irrelevant for astrophysical applications.

The following points will be discussed in detail:

- Solutions of Maxwell's equations in a Kerr background
- Space-time splittings
- The horizon as a conducting membrane
- Magnetic energy extraction from a BH
- BH's as current generators or rotators of electric motors
- The Blandford-Znajek process
- Stationary axisymmetric electrodynamics for force-free fields

1 Introduction

In these lectures I give an introduction to what is called *the membrane model of black holes* (BHs). This is not a new theory, but a convenient reformulation of the standard relativistic theory of BHs – as far as physics *outside* the horizon is concerned –, which

is much closer to the intuition we have gained from other fields of physics (see [1], for a general reference). While the basic equations look less elegant, it has the advantage that we can understand astrophysical processes near a BH much more easily. For an analogy, imagine you would have to explain how a Tokomak works by using the language and pictures of special relativity (SR), i.e., by using the electromagnetic field tensor and 4-dimensional pictures of plasma flows. I would not know how to do this and how to get, for instance, an understanding of even the simplest plasma instabilities. A closer analogy would be to translate relevant studies of the electrodynamics of pulsars into a 4-dimensional language. The basic equations look beautiful, but it would be hard to understand anything. (You may say that radio pulsars are anyhow not understood.)

In the membrane model (often called *membrane paradigm* [1]) one first splits the elegant 4-dimensional physical laws of general relativity (GR) into space and time (3+1 splitting). For a general situation this can be done in many ways (reflecting the gauge freedom in GR) since there is no canonical fibration of spacetime by level surfaces of constant time. However, for a stationary BH there is a preferred decomposition. Relative to this the dynamical variables (electromagnetic fields, etc) become quantities on an *absolute space* which evolve as functions of an *absolute time*, as we are accustomed to from nonrelativistic physics. We shall see, for example, that the 3+1 splitting brings Maxwell's equations into a form which resembles the familiar form of Maxwell's equations for moving conductors. We can then use the pictures and the experience from ordinary electrodynamics.

In a second step one replaces the boundary conditions at the horizon by physical properties (electric conductivity, etc) of a *fictitious membrane*. This procedure is completely adequate as long as one is not interested in fine details *very close* to the horizon. The details of this boundary layer are, however, completely irrelevant for astrophysical applications. (The situation is similar to many problems in electrodynamics, where one replaces the real surface properties of a conductor and other media by idealized boundary conditions.)

The program of these lectures is as follows. First I will discuss the 3+1 splitting of the spacetime of a stationary rotating BH and of Maxwell's equations outside its horizon. We shall see that this can be achieved very smoothly by using the calculus of differential forms. As an illustration and for later use we shall apply these tools for a discussion of an exact solution of Maxwell's equation on a Kerr background, which describes an asymptotically homogeneous magnetic field. We shall then derive the electromagnetic properties of the fictitious membrane that simulate the boundary conditions at the horizon. Here, I can offer a much simpler derivation than has been given so far in the literature. As an important example of a physical process relatively close to a BH I will treat in detail the magnetic energy extraction of a hole's rotational energy. Blandford and Znajek have first pointed out the possible relevance of this mechanism for an understanding of active galactic nuclei. It may well play an important role in the formation of energetic jets. The Blandford-Znajek process could also be important for explaining gamma-ray-bursts, because it may energize a Poynting-dominated outflow.

I hope to show you that the physics involved is not very different from that behind the electric generator in Fig. 1.

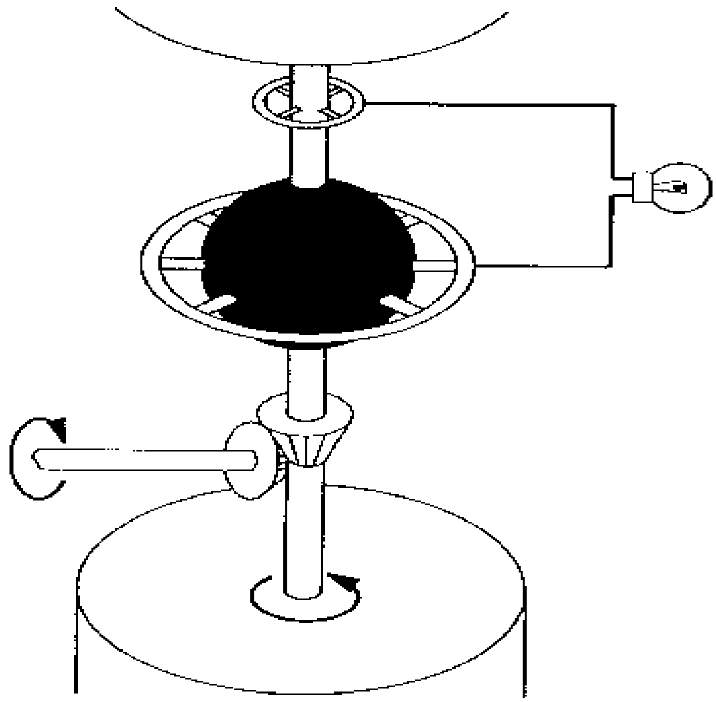

Figure 1: Electric generator whose physics is similar to the electrodynamics of black holes in external magnetic fields.

2 Space-Time Splitting of Electrodynamics

I describe now the 3+1 splitting of the general relativistic Maxwell equations on a stationary spacetime $(M, {}^{(4)}\boldsymbol{g})$. Most of what follows could easily be generalized to spacetimes which admit a foliation by spacelike hypersurfaces (see, e.g., Ref. [2]), but this is not needed in what follows.

Slightly more specifically, we shall assume that globally M is a product $\boldsymbol{R} \times \Sigma$, such that the natural coordinate t of $\boldsymbol{R}$ is adapted to the Killing field k, i.e., $k = \partial_t$. We decompose the Killing field into normal and parallel components relative to the "absolute space" $(\Sigma, \boldsymbol{g})$, $\boldsymbol{g}$ being the induced metric on Σ,

$$\partial_t = \alpha\, u + \beta. \tag{1}$$

Here u is the unit normal field and β is tangent to Σ. This is what one calls the decomposition into lapse and shift; α is the *lapse function* und β the *shift vector field*. We shall usually work with adapted coordinates $(x^\mu) = (t, x^i)$, where $\{x^i\}$ is a coordinate system on Σ. Let $\beta = \beta^i \partial_i \quad (\partial_i = \partial/\partial x^i)$, and consider the basis of 1-forms

$$\alpha\, dt, \qquad dx^i + \beta^i dt. \tag{2}$$

One verifies immediately, that this is dual to the basis $\{u, \partial_i\}$ of vector fields. Since u is perpendicular to the tangent vectors ∂_i of Σ, the 4-metric has the form

$${}^{(4)}\boldsymbol{g} = -\alpha^2 dt^2 + g_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right), \tag{3}$$

where $g_{ij}dx^i dx^j$ is the induced metric $\boldsymbol{g}$ on Σ. Clearly, α, β, and $\boldsymbol{g}$ are all time-independent quantities on Σ.

For what follows, I would like to change this setup slightly by using, instead of ∂_i and dx^i, a dual orthonormal pair $\{e_i\}$ and $\{\vartheta^i\}$ on Σ. Instead of (2), we have then the *orthonormal* tetrad

$$\theta^0 = \alpha dt, \qquad \theta^i = \vartheta^i + \beta^i dt, \tag{4}$$

where now $\beta = \beta^i e_i$. This is dual to the orthonormal frame

$$e_0 = u = \frac{1}{\alpha}(\partial_t - \beta), \quad e_i. \tag{5}$$

The tetrad $\{\theta^\mu\}$ describes the reference frames of so-called FIDOs, for *fiducial observers*. Their 4-velocity is thus perpendicular to the absolute space Σ.

Relative to these observers we have for the electromagnetic field tensor F (2-form) the same decomposition as in SR:

$$F = E \wedge \theta^0 + B, \tag{6}$$

where E is the electric 1-form $E = E_i \theta^i$ and B the magnetic 2-form $B = \frac{1}{2} B_{ij}\, \theta^i \wedge \theta^j$. ($E_i$, B_{ij} are the field strengths measured by the FIDOs.)

In a second step we decompose E and B relative to absolute space and absolute time. We have, using (5),

$$E = E_i \theta^i = E_i \left(\vartheta^i + \beta^i dt\right) = \mathcal{E} + i_\beta\, \mathcal{E}\, dt, \tag{7}$$

where

$$\mathcal{E} = E_i \vartheta^i, \qquad i_\bullet : \quad \text{interior product.} \tag{8}$$

Similarly,

$$B = \mathcal{B} + dt \wedge i_\beta\, \mathcal{B}, \quad \mathcal{B} = \frac{1}{2} B_{ij}\, \vartheta^i \wedge \vartheta^j. \tag{9}$$

Together we arrive at the following 3+1 decomposition of F:

$$F = \mathcal{B} + (\alpha\, \mathcal{E} - i_\beta\, \mathcal{B}) \wedge dt. \tag{10}$$

¿From this the 3+1 splitting of the homogeneous Maxwell equations is readily obtained: $dF = 0$ gives

$$\mathbf{d}\mathcal{B} + dt \wedge \partial_t\, \mathcal{B} + \mathbf{d}(\alpha\, \mathcal{E}) \wedge dt - \mathbf{d}(i_\beta\, \mathcal{B}) \wedge dt = 0.$$

Here $\mathbf{d}$ denotes the differential on Σ. This gives the two equations

$$\mathbf{d}\mathcal{B} = 0, \qquad \mathbf{d}(\alpha\, \mathcal{E}) + \partial_t\, \mathcal{B} = \mathbf{d}(i_\beta\, \mathcal{B})$$

or, with the Cartan identity $L_\beta = \mathbf{d} \circ i_\beta + i_\beta \circ \mathbf{d}$,

$$\mathbf{d}\mathcal{B} = 0, \qquad \mathbf{d}(\alpha\, \mathcal{E}) + (\partial_t - L_\beta)\mathcal{B} = 0. \tag{11}$$

The second equation describes Faraday's induction law in a gravitational field. It will be of crucial importance in later sections. Note, in particular, the coupling of the $\mathcal{B}$-field to the shift through the Lie derivative.

Let us also decompose the representation of F by a potential, $F = dA$. We have, using again (4),

$$\begin{aligned} A &= A_\mu \theta^\mu = \alpha A_0\, dt + A_i(\vartheta^i + \beta^i\, dt) \\ &= (\alpha A_0 + i_\beta\, \mathcal{A})dt + \mathcal{A}, \qquad \mathcal{A} = A_i\, \vartheta^i. \end{aligned}$$

Thus

$$A = -\phi\, dt + \mathcal{A}, \tag{12}$$

where

$$\phi = -(\alpha A_0 + i_\beta\, \mathcal{A}). \tag{13}$$

This gives

$$dA = -\mathbf{d}\phi \wedge dt + \mathbf{d}\mathcal{A} + dt \wedge \partial_t\, \mathcal{A},$$

which is of the form (10), with

$$\mathcal{B} = \mathbf{d}\mathcal{A}, \qquad \alpha\, \mathcal{E} = -\mathbf{d}\phi - \partial_t\, \mathcal{A} + i_\beta\, \mathbf{d}\mathcal{A}. \tag{14}$$

Apart from the last term, this is what one is used to.

Now, we turn to the inhomogeneous Maxwell equation

$$d * F = 4\pi \mathcal{S}. \tag{15}$$

We need first the Hodge-dual of (10). We decompose $*F$ similarly to (6):

$$*F = -H \wedge \theta^0 + D, \tag{16}$$

which can be viewed as a definition of H and D. Comparison with (6) shows, that

$$\begin{aligned} H &= B_i\, \theta^i, \qquad B_1 = B_{23}, \text{ etc}, \\ D &= E_1\, \theta^2 \wedge \theta^3 + E_2\, \theta^3 \wedge \theta^1 + E_3\, \theta^1 \wedge \theta^2. \end{aligned} \tag{17}$$

With (4) we find ($*$ denotes the Hodge-dual on Σ)

$$\begin{aligned} H &= \mathcal{H} + i_\beta\, \mathcal{H} \wedge dt, \qquad \mathcal{H} = *\mathcal{B}, \\ D &= \mathcal{D} - i_\beta\, \mathcal{D} \wedge dt, \qquad \mathcal{D} = *\mathcal{E}. \end{aligned} \tag{18}$$

If this is inserted into (16), we obtain

$$*F = \mathcal{D} - (\alpha \mathcal{H} + i_\beta\, \mathcal{D}) \wedge dt. \tag{19}$$

The dual $J = *\mathcal{S}$ of the current 3-form can be decomposed as in SR

$$J = \rho_{el}\, \theta^0 + j_k\, \theta^k, \tag{20}$$

where ρ_{el} is the electric charge density and j^k is the electric current density relative to the FIDOs. We use them to introduce the following quantities on the absolute space

$$\rho = \rho_{el}\,\vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3, \qquad j = j_k\,\vartheta^k, \qquad \mathcal{J} = *j. \tag{21}$$

Using the notation $\eta^\mu := *\theta^\mu$, we can decompose $\mathcal{S}$ as follows

$$\begin{aligned}
\mathcal{S} &= \rho_{el}\,\eta^0 + j_k\,\eta^k \\
&= \rho_{el}\,\theta^1 \wedge \theta^2 \wedge \theta^3 - (j_1\,\theta^2 \wedge \theta^3 + \dots) \wedge \theta^0 \\
&= \rho + \rho_{el}(\beta^1\,\vartheta^2 \wedge \vartheta^3 + \dots) \wedge dt - \alpha\,(j_1\,\vartheta^2 \wedge \vartheta^3 + \dots) \wedge dt.
\end{aligned}$$

Thus

$$\mathcal{S} = \rho + (i_\beta\,\rho - \alpha\,\mathcal{J}) \wedge dt. \tag{22}$$

Inserting this and (19) into (15) leads to

$$\begin{aligned}
d * F &= \mathbf{d}\mathcal{D} + dt \wedge \partial_t\,\mathcal{D} - \mathbf{d}(\alpha\mathcal{H}) \wedge dt - \mathbf{d}(i_\beta\,\mathcal{D}) \wedge dt \\
&= 4\pi\rho + 4\pi(i_\beta\,\rho - \alpha\,\mathcal{J}) \wedge dt,
\end{aligned}$$

and hence to the following 3+1 split of the inhomogeneous Maxwell equation

$$\mathbf{d}\mathcal{D} = 4\pi\rho, \qquad \mathbf{d}(\alpha\mathcal{H}) = (\partial_t - L_\beta)\mathcal{D} + 4\pi\alpha\,\mathcal{J}. \tag{23}$$

¿From these laws one obtains immediately the local conservation law of the electric charge (use that $\mathbf{d}$ commutes with L_β):

$$(\partial_t - L_\beta)\rho + \mathbf{d}(\alpha\,\mathcal{J}) = 0. \tag{24}$$

This follows, of course, also from $d\mathcal{S} = 0$ and the decomposition (22).

Integral Formulas

As is well-known from ordinary electrodynamics, it is often useful to write the basic laws (11), (23), and (24) in integral forms. Consider, for instance, the induction law in (11). If we integrate this over a surface area $\mathcal{A}$, which is *at rest* relative to the absolute space, we obtain with Stokes' theorem ($\mathcal{C} := \partial\mathcal{A}$)

$$\oint_{\mathcal{C}} \alpha\,\mathcal{E} = -\frac{d}{dt}\int_{\mathcal{A}} \mathcal{B} + \int_{\mathcal{A}} L_\beta\,\mathcal{B}.$$

Here, we use $L_\beta\,\mathcal{B} = \mathbf{d}\,i_\beta\,\mathcal{B}$ (since $\mathbf{d}\mathcal{B} = 0$) and Stokes' theorem once more, with the result

$$\oint_{\mathcal{C}} \alpha\,\mathcal{E} = -\frac{d}{dt}\int_{\mathcal{A}} \mathcal{B} + \oint_{\mathcal{C}} i_\beta\,\mathcal{B}. \tag{25}$$

The left hand side is the electromotive force (EMF) along $\mathcal{C}$. The last term is similar to the additional term one encounters in Faraday's induction law for moving conductors. It is an expression of the coupling of $\mathcal{B}$ to the gravitomagnetic field and plays a crucial

role in much that follows. This term contributes also for a stationary situation, for which (25) reduces to

$$\mathrm{EMF}(\mathcal{C}) = \oint_{\mathcal{C}} \alpha\,\mathcal{E} = \oint_{\mathcal{C}} i_\beta\,\mathcal{B}. \tag{26}$$

The integral form of the Ampère-Maxwell law is obtained similarly. Integrating the second equation in (23), we obtain with the Cartan identity $L_\beta = \mathbf{d} \circ i_\beta + i_\beta \circ \mathbf{d}$ and Gauss' law (first equation in (23)):

$$\oint_{\mathcal{C}} (\alpha\,\mathcal{H} + i_\beta\,\mathcal{D}) = \frac{d}{dt}\int_{\mathcal{A}} \mathcal{D} + 4\pi \int_{\mathcal{A}} (\alpha\,\mathcal{J} - i_\beta\,\rho). \tag{27}$$

The integral form of charge conservation is obtained by integrating (24) over a volume $\mathcal{V}$ which is at rest relative to absolute space:

$$\frac{d}{dt}\int_{\mathcal{V}} \rho = -\int_{\partial\mathcal{V}} (\alpha\,\mathcal{J} - i_\beta\,\rho) \tag{28}$$

(note that $L_\beta\,\rho = \mathbf{d}\,i_\beta\,\rho$).

One could, of course, also derive integral formulas for moving volumes and surface areas (exercise).

Vector Analytic Formulation

The similarity of the basic laws in the 3+1 split with ordinary electrodynamics becomes even closer if we write everything in vector analytic form. I give a dictionary between the two formulations that is valid for any 3-dimensional Riemannian manifold $(\Sigma, \boldsymbol{g})$.

The metric $\boldsymbol{g}$ defines natural isomorphisms $\sharp$ and $\flat$ between the sets of 1-forms, $\Lambda^1(\Sigma)$, and vector fields, $\mathcal{X}(\Sigma)$. In addition, the volume form η, belonging to the metric, defines an isomorphism between $\mathcal{X}(\Sigma)$ and the space of 2-forms, $\Lambda^2(\Sigma)$, given by

$$\vec{B} \longmapsto \mathcal{B} = i_{\vec{B}}\,\eta. \tag{29}$$

We have the following commutative diagram, in which $*$ denotes, as always, the Hodge-dual:

$$\begin{tikzcd}
\Lambda^1(\Sigma) \arrow[rr, shift left, "*"] \arrow[dr, shift right, "\sharp"'] & & \Lambda^2(\Sigma) \arrow[ll, shift left, "*"] \\
& \mathcal{X}(\Sigma) \arrow[ul, shift right, "\flat"'] \arrow[ur, "i_\bullet\,\eta"'] &
\end{tikzcd}$$

¿From this one can read off, for instance,

$$i_{\vec{v}}\,\eta = *v \qquad (\vec{v} \in \mathcal{X}(\Sigma),\ v:\ (\vec{v})^\flat). \tag{30}$$

The cross product and the wedge product are related as follows:

$$\begin{array}{ccc}
\Lambda^1(\Sigma) \times \Lambda^1(\Sigma) & \xrightarrow{\wedge} & \Lambda^2(\Sigma) \\
\updownarrow & & \updownarrow \\
\mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) & \xrightarrow{\times} & \mathcal{X}(\Sigma)
\end{array}$$

In particular, we have

$$i_{\vec{v}\times\vec{w}}\,\eta = v \wedge w. \tag{31}$$

With the help of the next commutative diagram one can reduce many of the vector analytic identities to $d \circ d = 0$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda^0(\Sigma) & \xrightarrow{\mathbf{d}} & \Lambda^1(\Sigma) & \xrightarrow{\mathbf{d}} & \Lambda^2(\Sigma) & \xrightarrow{\mathbf{d}} & \Lambda^3(\Sigma) & \longrightarrow & 0 \\
 & & \| & & \flat\uparrow\downarrow\sharp & & \uparrow i_{\bullet}\,\eta & & \uparrow \bullet\,\eta & & \\
0 & \longrightarrow & \Lambda^0(\Sigma) & \xrightarrow{\mathrm{grad}} & \mathcal{X}(\Sigma) & \xrightarrow{\mathrm{curl}} & \mathcal{X}(\Sigma) & \xrightarrow{\mathrm{div}} & \Lambda^0(\Sigma) & \longrightarrow & 0.
\end{array}$$

We can read off, for example,

$$i_{\mathrm{curl}\,\vec{v}}\,\eta = \mathbf{d}v. \tag{32}$$

For a 1-form w with vector field $\vec{w} = w^\sharp$,we have the translation

$$i_{\vec{v}}\,\mathbf{d}w \overset{(32)}{=} i_{\mathrm{curl}\,\vec{v}}\, i_{\mathrm{curl}\,\vec{w}}\,\eta = [(\mathrm{curl}\,\vec{w}) \times \vec{v}]^\flat. \tag{33}$$

Here, we made use of the algebraic relation

$$i_{\vec{v}}\, i_{\vec{u}}\,\eta \overset{(30)}{=} i_{\vec{v}} * u = *(u \wedge v) \overset{(31)}{=} (\vec{u} \times \vec{v})^\flat. \tag{34}$$

We need also the translation of the Lie derivative of a 1-form w:

$$L_{\vec{v}}\,w = \mathbf{d}\,\underbrace{i_{\vec{v}}\,w}_{(\vec{v},\vec{w})} + i_{\vec{v}}\,\mathbf{d}w \overset{(33)}{=} \mathbf{d}(\vec{v},\vec{w}) + [(\mathrm{curl}\,\vec{w}) \times \vec{v}]^\flat.$$

Thus,

$$L_{\vec{v}}\,w = \{\mathrm{grad}\,(\vec{v},\vec{w}) + (\mathrm{curl}\,\vec{w}) \times \vec{v}\}^\flat. \tag{35}$$

Similarly, we have for a 2-form $\mathcal{B} = i_{\vec{B}}\,\eta$:

$$\begin{aligned}
L_{\vec{v}}\,\mathcal{B} &= L_{\vec{v}}\, i_{\vec{B}}\,\eta = i_{\vec{v}}\,\underbrace{\mathbf{d}\, i_{\vec{B}}\,\eta}_{\mathrm{div}\,\vec{B}\,\eta} + \mathbf{d}\,\underbrace{i_{\vec{v}}\, i_{\vec{B}}\,\eta}_{(\vec{B}\times\vec{v})^\flat} \\
&\overset{(32)}{=} i_{\{(\mathrm{div}\,\vec{B})\vec{v} + \mathrm{curl}\,(\vec{B}\times\vec{v})\}}\,\eta.
\end{aligned}$$

We have thus the correspondence

$$L_{\vec{v}}\,\mathcal{B} \longleftrightarrow (\mathrm{div}\,\vec{B})\vec{v} \;+\; \mathrm{curl}\,(\vec{B} \times \vec{v}). \tag{36}$$

Here we have to stress that the right hand side is in general not equal to $L_{\vec{v}}\,\vec{B} = [\vec{v},\vec{B}]$ (Lie bracket). This comes out as follows:

$$\begin{aligned}
L_{\vec{v}}\,\mathcal{B} &= L_{\vec{v}}\, i_{\vec{B}}\,\eta = \underbrace{[L_{\vec{v}}, i_{\vec{B}}]}_{i_{[\vec{v},\vec{B}]}}\,\eta + i_{\vec{B}}\,\underbrace{L_{\vec{v}}\,\eta}_{\mathrm{div}\,\vec{v}\,\eta} \\
&= i_{\left\{(\mathrm{div}\,\vec{v})\vec{B} + [\vec{v},\vec{B}]\right\}}\,\eta.
\end{aligned}$$

The correspondence (36) is thus equivalent to

$$L_{\vec{v}}\,\mathcal{B} \longleftrightarrow L_{\vec{v}}\,\vec{B} + (\mathrm{div}\,\vec{v})\,\vec{B}. \tag{37}$$

Only for $\operatorname{div}\vec{v}=0$ do the Lie derivatives $L_{\vec{v}}\mathcal{B}$ and $L_{\vec{v}}\vec{B}$ correspond to each other!

In Maxwell's equations the Lie derivative $L_{\vec{\beta}}\mathcal{B}$ occurs. This will be replaced in the vector analytic translation by $L_{\vec{\beta}}\vec{B}$, because $\operatorname{div}\vec{\beta}$ is (for a stationary metric) proportional to the trace of the second fundamental form of the time slices and this vanishes for *maximal slicing*. For the Kerr solution we are, for instance, in this situation (exercise).

Part of what has been said is summarized for convenience in the table below.

Dictionary

calculus of forms	vector analysis	notation
$v\wedge w$	$\vec{v}\times\vec{w}$	$v=(\vec{v})^\flat,\ w=(\vec{w})^\flat$
$i_{\vec{v}}\mathcal{B},\ i_{\vec{v}}\mathcal{D}$	$\vec{B}\times\vec{v},\ \vec{E}\times\vec{v}$	$\mathcal{B}=i_{\vec{B}}\eta,\ \mathcal{D}=i_{\vec{E}}\eta$
$\mathbf{d}f$	$\operatorname{grad}f$	$f:$ function
$\mathbf{d}v$	$\operatorname{curl}\vec{v}$	
$\mathbf{d}\mathcal{B},\ \mathbf{d}\mathcal{D}$	$\operatorname{div}\vec{B},\ \operatorname{div}\vec{E}$	
$L_{\vec{v}}w$	$\operatorname{grad}(\vec{v},\vec{w})-\vec{v}\times\operatorname{curl}\vec{w}$	$w=(\vec{w})^\flat$
$L_{\vec{v}}\mathcal{B},\ L_{\vec{v}}\mathcal{D}$	$(\operatorname{div}\vec{B})\vec{v}-\operatorname{curl}(\vec{v}\times\vec{B}),\ \vec{B}\longleftrightarrow\vec{E}$	

Summary

For reference, we write down once more the 3+1 split of Maxwell's equations (11) and (23) in Cartan's calculus

$$\begin{aligned} \mathbf{d}\mathcal{B}&=0, &\quad \mathbf{d}(\alpha\,\mathcal{E})+(\partial_t-L_\beta)\mathcal{B}&=0,\\ \mathbf{d}\mathcal{D}&=4\pi\rho, &\quad \mathbf{d}(\alpha\mathcal{H})&=(\partial_t-L_\beta)\mathcal{D}+4\pi\alpha\,\mathcal{J}. \end{aligned} \tag{38}$$

The dictionary above allows us to translate these into the vector analytic form:

$$\begin{aligned} \vec{\nabla}\cdot\vec{B}&=0, &\quad \vec{\nabla}\times(\alpha\,\vec{E})+(\partial_t-L_{\vec{\beta}})\vec{B}&=0,\\ \vec{\nabla}\cdot\vec{E}&=4\pi\rho_{el}, &\quad \vec{\nabla}\times(\alpha\,\vec{B})&=(\partial_t-L_{\vec{\beta}})\vec{E}+4\pi\alpha\,\vec{j}. \end{aligned} \tag{39}$$

3 Black Hole in a Homogeneous Magnetic Field

As an instructive example and a useful tool we discuss now an exact solution of Maxwell's equations in the Kerr metric, which becomes asymptotically a homogeneous magnetic field. This solution can be found in a strikingly simple manner [3].

For any Killing field K one has the following identity

$$\delta\,d\,K^\flat=2\,R(K), \tag{40}$$

where δ denotes the codifferential and $R(K)$ is the 1-form with components $R_{\mu\nu}K^\nu$. In components (40) is equivalent to

$$K_{\mu\ ;\alpha}^{\ ;\alpha}=-R_{\mu\alpha}\,K^\alpha. \tag{41}$$

This form can be obtained by contracting the indices σ and ρ in the following general equation for a vector field

$$\xi_{\sigma;\rho\mu}-\xi_{\sigma;\mu\rho}=\xi_\lambda R^\lambda_{\sigma\rho\mu}$$

and by using the consequence $K^{\sigma}_{;\sigma} = 0$ of the Killing equation $K_{\sigma;\rho} + K_{\rho;\sigma} = 0$.

For a vacuum spacetime we thus have

$$\delta\, d\, K^{\flat} = 0 \tag{42}$$

for any Killing field. Hence, the vacuum Maxwell equations are satisfied if F is a constant linear combination of the differential of Killing fields (their duals, to be precise). For the Kerr metric, as for any axially symmetric stationary spacetime, we have two Killing fields k and m, say; in adapted coordinates these are $k = \partial_t$ and $m = \partial_\varphi$. The Komar formulae provide convenient expressions for the total mass M and the total angular momentum J of the Kerr BH:

$$M = -\frac{1}{8\pi}\int_\infty *dk^\flat, \qquad J = \frac{1}{16\pi}\int_\infty *dm^\flat \tag{43}$$

(for $G = 1$).

We try the ansatz

$$F = \frac{1}{2} B_0 \,(dm^\flat + 2a\, dk^\flat) \qquad (B_0 = \text{const}), \tag{44}$$

and choose a such that the total electric charge

$$Q = -\frac{1}{4\pi}\int_\infty *F \tag{45}$$

vanishes. The Komar formulae (43) tell us that

$$Q = -\frac{1}{8\pi} B_0 \,(16\pi J - 2a \cdot 8\pi M), \tag{46}$$

and this vanishes if $a = J/M$ (which is the standard meaning of the symbol a in the Kerr solution).

Clearly, F is stationary and axisymmetric:

$$L_k\, F = L_m\, F = 0, \tag{47}$$

because (dropping $\flat$ from now on)

$$L_k\, dk = d\, L_k\, k = 0 \qquad (L_k\, k = [k,k] = 0), \text{ etc.}$$

For the further discussion we need the Kerr metric. In Boyer-Lindquist coordinates and more or less standard notation it has the form (3), i.e.,

$${}^{(4)}\boldsymbol{g} = [-\alpha^2\, dt^2 + g_{\varphi\varphi}(d\varphi + \beta^\varphi\, dt)^2] + [g_{rr}\, dr^2 + g_{\vartheta\vartheta}\, d\vartheta^2], \tag{48}$$

with only the component β^φ of the shift being $\neq 0$. With the abbreviations

$$\rho^2 := r^2 + a^2\cos^2\vartheta, \quad \Delta := r^2 - 2Mr + a^2, \quad \Sigma^2 := (r^2+a^2)^2 - a^2\Delta\sin^2\vartheta, \tag{49}$$

the metric coefficients are

$$\begin{aligned} g_{rr} &= \frac{\rho^2}{\Delta}, \quad g_{\vartheta\vartheta} = \rho^2, \qquad g_{\varphi\varphi} = \sin^2\vartheta\,\frac{\Sigma^2}{\rho^2}, \\ g_{tt} &= -1 + \frac{2Mr}{\rho^2}, \qquad g_{t\varphi} = -\frac{2Mra\sin^2\vartheta}{\rho^2}, \end{aligned} \tag{50}$$

while the lapse and shift are given by

$$\alpha^2 = \frac{\rho^2}{\Sigma^2}\,\Delta, \qquad \beta^\varphi = -a\,\frac{2Mr}{\Sigma^2}. \tag{51}$$

This gives asymptotically

$$\begin{aligned} {}^{(4)}\boldsymbol{g} \;&=\; -\left[1 - \frac{2M}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] dt^2 - \left[\frac{4aM}{r}\,\sin^2\vartheta + \mathcal{O}\left(\frac{1}{r^2}\right)\right] dt\,d\varphi \\ &\quad + \left[1 + \mathcal{O}\left(\frac{1}{r}\right)\right]\,\left[dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2)\right]. \end{aligned} \tag{52}$$

To establish the connection with our general discussion, we introduce here as orthonormal basis of the absolute space naturally

$$\vartheta^r = \sqrt{g_{rr}}\,dr, \quad \vartheta^\vartheta = \sqrt{g_{\vartheta\vartheta}}\,d\vartheta, \quad \vartheta^\varphi = \sqrt{g_{\varphi\varphi}}\,d\varphi. \tag{53}$$

The shift vector is $\beta = \beta^\varphi\,\partial_\varphi$.

The angular velocity ω of the FIDOs, with 4-velocity $u = \frac{1}{\alpha}(\partial_t - \beta^\varphi\,\partial_\varphi)$, is

$$\omega = \frac{u^\varphi}{u^t} = -\beta^\varphi = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}. \tag{54}$$

The last equality sign implies that the angular momentum of these (Bardeen) observers vanishes: $(u, m) = 0$. The FIDOs are, therefore, sometimes also called ZAMOs (for *zero angular momentum observers*).

Now, we want to discuss in detail the solution (44), which we can express in terms of a potential: $F = dA$, with

$$A = \frac{1}{2}\,B_0\,(m + 2ak). \tag{55}$$

Let us first look at the asymptotics. The 1-forms $k_\mu\,dx^\mu$, $m_\mu\,dx^\mu$ belonging to the Killing fields ($k_\mu = g_{\mu t}$, $m_\mu = g_{\mu\varphi}$) are asymptotically $k \sim -dt$, $m \sim r^2\sin^2\vartheta\,d\varphi$, whence (44) gives

$$F \sim B_0\,[\sin\vartheta\,dr \wedge r\sin\vartheta\,d\varphi + \cos\vartheta\,r d\vartheta \wedge r\sin\vartheta\,d\varphi]. \tag{56}$$

This is a magnetic field in the z-direction whose magnitude is B_0.

In (55) we need

$$\begin{aligned} m + 2ak \;&=\; (g_{\mu\varphi} + 2a\,g_{\mu t})\,dx^\mu = (g_{t\varphi} + 2a\,g_{tt})\,dt + (g_{\varphi\varphi} + 2a\,g_{\varphi t})\,d\varphi \\ &\overset{(54)}{=}\; (-\omega\,g_{\varphi\varphi} + 2a\,g_{tt})dt + (g_{\varphi\varphi} - 2a\,\omega\,g_{\varphi\varphi})d\varphi. \end{aligned}$$

Using the notation [1]

$$\tilde\omega^2 := g_{\varphi\varphi}, \qquad \tilde\omega = \frac{\Sigma}{\rho}\,\sin\vartheta \tag{57}$$

we thus have

$$m + 2ak = \left[-\omega\tilde\omega^2 + 2a\,\left(\omega^2\tilde\omega^2 - \alpha^2\right)\right]\,dt + \tilde\omega^2(1 - 2a\omega)\,d\varphi.$$

Comparing this with (12), we obtain for the potentials

$$\phi = \frac{1}{2} B_0 \left[\omega\tilde{\omega}^2 + 2a\left(\alpha^2 - \omega^2\tilde{\omega}^2\right)\right], \tag{58}$$

$$\mathcal{A} = \mathcal{A}_\varphi \, d\varphi, \qquad \mathcal{A}_\varphi = \frac{1}{2} B_0 \, \tilde{\omega}^2 (1 - 2a\omega). \tag{59}$$

The fields $\mathcal{E}$ and $\mathcal{B}$ can now be obtained from (14). We have

$$\begin{aligned} \mathcal{B} &= d\mathcal{A} = \mathcal{A}_{\varphi,r}\, dr \wedge d\varphi + \mathcal{A}_{\varphi,\vartheta}\, d\vartheta \wedge d\varphi \\ &= \frac{1}{\Sigma \sin\vartheta} \left[\mathcal{A}_{\varphi,\vartheta} \underbrace{\vartheta^2 \wedge \vartheta^3}_{*\vartheta^1} - \sqrt{\Delta}\, \mathcal{A}_{\varphi,r} \underbrace{\vartheta^3 \wedge \vartheta^1}_{*\vartheta^2} \right]. \end{aligned} \tag{60}$$

$\mathcal{A}_\varphi$ is explicitly (use (59), (57), (54), and (51))

$$\mathcal{A}_\varphi = \frac{1}{2} B_0 \frac{\Sigma^2}{\rho^2} \sin^2\vartheta \left(1 - 4a^2 \frac{Mr}{\Sigma^2}\right).$$

We write this as

$$\mathcal{A}_\varphi = \frac{1}{2} B_0 \, X, \qquad X = \frac{\sin^2\vartheta}{\rho^2} (\Sigma^2 - 4a^2 Mr). \tag{61}$$

With this notation, (60) reads

$$\mathcal{B} = \frac{B_0}{2\Sigma \sin\vartheta} \left[X_{,\vartheta} * \vartheta^r - \sqrt{\Delta}\, X_{,r} * \vartheta^\vartheta \right]. \tag{62}$$

The corresponding vector field $\vec{B}$ is thus

$$\vec{B} = \frac{B_0}{2\Sigma \sin\vartheta} \left[X_{,\vartheta}\, \vec{e}_r - \sqrt{\Delta}\, X_{,r}\, \vec{e}_\vartheta \right]. \tag{63}$$

For $\mathcal{E}$ we have, with $\beta = -\omega\, \partial_\varphi$,

$$\alpha\, \mathcal{E} = -\mathbf{d}\phi + i_\beta\, \mathbf{d}\mathcal{A} = -\mathbf{d}\phi + \omega\, \mathbf{d}\mathcal{A}_\varphi. \tag{64}$$

¿From this one finds quickly

$$\begin{aligned} \vec{E} = &-\frac{B_0\, a\Sigma}{\rho^2} \left\{ \left[\frac{\partial\alpha^2}{\partial r} + \frac{M \sin^2\vartheta}{\rho^2} (\Sigma^2 - 4a^2 Mr) \frac{\partial}{\partial r}\left(\frac{r}{\Sigma^2}\right) \right] \vec{e}_r \right. \\ &\left. + \frac{1}{\sqrt{\Delta}} \left[\frac{\partial\alpha^2}{\partial\vartheta} + r\, \frac{M \sin^2\vartheta}{\rho^2} (\Sigma^2 - 4a^2 Mr) \frac{\partial}{\partial\vartheta}\left(\frac{1}{\Sigma^2}\right) \right] \vec{e}_\vartheta \right\}. \end{aligned} \tag{65}$$

The field lines of $\vec{E}$ are shown in Fig. 2.

It is of interest to work out the magnetic flux through the equator of the BH, i.e.,

$$\Phi = \int_{\text{upper h.}} \mathcal{B} = \int_{\text{equator}} \mathcal{A} = 2\pi\, \mathcal{A}_\varphi \,|_{\text{equator}}\,. \tag{66}$$

Specializing (61) to $\vartheta = \pi/2$ and $r = r_H$ gives (the horizon is located at $\Delta = 0$)

$$\Phi = 4\pi B_0\, M(r_H - M) = 4\pi B_0\, M\sqrt{M^2 - a^2}. \tag{67}$$

Here, we have used

$$r_H = M + \sqrt{M^2 - a^2}. \tag{68}$$

Note that this vanishes for an extremal BH ($a = M$). Generically one has, as expected, $\Phi \approx \pi r_H^2 B_0$.

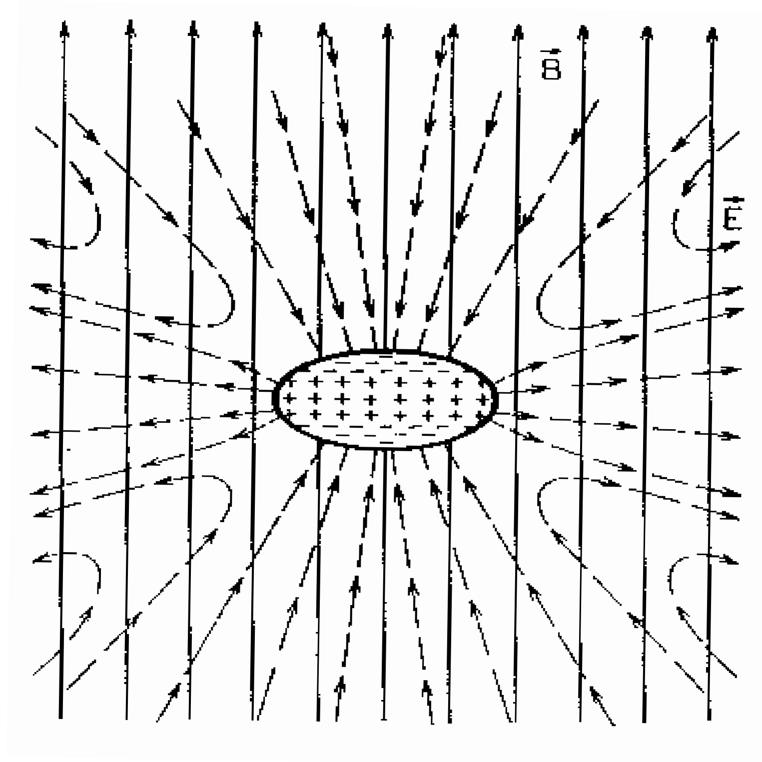

Figure 2: Field lines of $\vec{E}$ (from Ref. [1]).

4 The Horizon as a Conducting Membrane

As long as one is not interested in fine details very close to the horizon, one can regard the boundary conditions implied by the horizon as arising from physical properties of a fictitious membrane. This membrane is thought of as endowed with surface charge density σ_H, surface current $\vec{\mathcal{J}}_H$, and surface resistivity R_H. This idea was first pursued by Damour [4] and independently by Znajek [5]. Later it was further developed by Thorne and Mac Donald [6] and other authors (see [1] and references therein).

Below we give a much simplified derivation that the following surface properties for the parallel and perpendicular components hold:

$$\begin{aligned} \text{Gauss's law}: &\quad E_\perp && \longrightarrow 4\pi\,\sigma_H, \\ \text{Ampère's law}: &\quad \alpha\,\vec{B}_\parallel && \longrightarrow \vec{B}_H = 4\pi\,\vec{\mathcal{J}}_H \times \vec{n}, \\ \text{charge conservation}: &\quad \alpha\,j_\perp && \longrightarrow -\partial_t\,\sigma_H - {}^{(2)}\vec{\nabla}\cdot\vec{\mathcal{J}}_H, \\ \text{Ohm's law}: &\quad \alpha\,\vec{E}_\parallel && \longrightarrow \vec{E}_H = R_H\,\vec{\mathcal{J}}_H. \end{aligned} \tag{69}$$

The surface resistivity R_H turns out to be equal to the vacuum impedance

$$R_H = 4\pi = 377 \text{ Ohm}. \tag{70}$$

For the horizon fields $\vec{E}_H$, $\vec{B}_H$ we have, therefore,

$$\vec{B}_H = \vec{E}_H \times \vec{n}, \tag{71}$$

as for a plane wave in vacuum.

Toward the horizon the FIDOs move relative to freely falling observers, say, more and more rapidly, approaching the velocity of light. Mathematically, the tetrad $\{\theta^\mu\}$

becomes singular at the horizon, and therefore the components of F relative to $\{\theta^\mu\}$ are ill-behaved. The 2-form F should, of course, remain regular and the laws (69) and (70) are just an expression of this requirement. For illustration, we demonstrate this first for a Schwarzschild BH, and generalize afterwards the argument in a simple manner.

(a) Derivation for a Schwarzschild BH

The procedure is simple: we pass to a regular, not necessarily orthonormal tetrad. The angular part $\theta^2 = r\,d\vartheta$, $\theta^3 = r\sin\vartheta\,d\varphi$ is kept, but instead of $\theta^0 = \alpha\,dt$, $\theta^1 = \alpha^{-1}\,dr$ we use $d\bar{t}$, where $\bar{t}$ is the Eddington-Finkelstein time coordinate

$$\bar{t} = t + 2M\,\ln\left(\frac{r}{2M} - 1\right) \longrightarrow d\bar{t} = dt + \alpha^{-2}\,\frac{2M}{r}\,dr \qquad (r > 2M). \tag{72}$$

Since the Eddington-Finkelstein coordinates are regular at the horizon, the same is true for the basis $\{d\bar{t},\, dr,\, \theta^2,\, \theta^3\}$. We have the relations

$$\begin{aligned} \theta^0 &= \alpha\,d\bar{t} - \frac{1}{\alpha}\frac{2M}{r}\,dr, \\ \theta^1 &= \frac{1}{\alpha}\,dr, \end{aligned} \tag{73}$$

which allow us to rewrite the decomposition (6) as follows

$$\begin{aligned} F &= E_1\,\theta^1\wedge\theta^0 + \ldots + B_3\,\theta^1\wedge\theta^2 + \ldots \\ &= E_1\,dr\wedge d\bar{t} + E_2\left(\alpha\,\theta^2\wedge d\bar{t} - \frac{1}{\alpha}\frac{2M}{r}\,\theta^2\wedge dr\right) + E_3\,(\ldots) \\ &\quad + B_3\,\frac{1}{\alpha}\,dr\wedge\theta^2 - B_2\,\frac{1}{\alpha}\,dr\wedge\theta^3 + B_1\,\theta^2\wedge\theta^3 \end{aligned}$$

or

$$\begin{aligned} F &= E_1\,dr\wedge d\bar{t} + \alpha\,E_2\,\theta^2\wedge d\bar{t} + \alpha\,E_3\,\theta^3\wedge d\bar{t} + B_1\,\theta^2\wedge\theta^3 \\ &\quad + \frac{1}{\alpha}\left(B_3 + \frac{2M}{r}\,E_2\right)dr\wedge\theta^2 + \frac{1}{\alpha}\left(-B_2 + \frac{2M}{r}\,E_3\right)dr\wedge\theta^3. \end{aligned} \tag{74}$$

The regularity of the coefficients in this expansion implies the following behavior when the horizon is approached ($\alpha \downarrow 0$):

$$\begin{aligned} &(i) \quad \text{radial components:} \quad E_1,\ B_1 = \mathcal{O}(1), \\ &(ii) \quad \text{tangent components:} \quad E_2,\ E_3,\ B_2,\ B_3 = \mathcal{O}\left(\frac{1}{\alpha}\right), \\ &(iii) \quad E_2 + B_3,\ E_3 - B_2 \quad = \mathcal{O}(\alpha). \end{aligned} \tag{75}$$

This shows that

$$\begin{aligned} \alpha\,\vec{E}_\parallel &= \vec{n}\times\alpha\,\vec{B}_\parallel + \mathcal{O}(\alpha^2), \qquad (\vec{n} := \vec{e}_r), \\ \alpha\,\vec{B}_\parallel &= -\vec{n}\times\alpha\,\vec{E}_\parallel + \mathcal{O}(\alpha^2), \end{aligned} \tag{76}$$

$$E_n \equiv E_\perp, \qquad B_n \equiv B_\perp \text{ remain finite.} \tag{77}$$

This is basically already what was claimed in (69) and (70). It is useful to introduce the *stretched horizon* $\mathcal{H}^s = \{\alpha = \alpha_H \ll 1\}$ which is arbitrary close to the event horizon. Let

$$\vec{E}_H := (\alpha\,\vec{E}_\parallel)_{\alpha_H}, \qquad \vec{B}_H := (\alpha\,\vec{B}_\parallel)_{\alpha_H} \tag{78}$$

and as above

$$E_n := \vec{E}\cdot\vec{n}, \qquad B_n := \vec{B}\cdot\vec{n}. \tag{79}$$

These components remain *finite* for $\alpha_H \downarrow 0$ and we have up to $\mathcal{O}(\alpha_H^2)$

$$\vec{E}_H = \vec{n}\times\vec{B}_H, \qquad \vec{B}_H = -\vec{n}\times\vec{E}_H. \tag{80}$$

The surface charge density σ_H and the surface current density $\vec{\mathcal{J}}_H$ on $\mathcal{H}^s$ are defined by

$$\sigma_H := \left(\frac{E_n}{4\pi}\right)_{\mathcal{H}^s}, \qquad \vec{B}_H =: \left(4\pi\,\vec{\mathcal{J}}_H\times\vec{n}\right)_{\mathcal{H}^s}. \tag{81}$$

The second equation and (50) imply Ohm's law in (69)

$$\vec{\mathcal{J}}_H = \frac{1}{R_H}\,\vec{E}_H, \qquad R_H = 4\pi. \tag{82}$$

Finally, we make use of the Ampère-Maxwell law (39) (for $\vec{\beta}=0$):

$$\partial_t\,\vec{E} = \vec{\nabla}\times(\alpha\,\vec{B}) - 4\pi\,\alpha\,\vec{j}.$$

The normal component on $\mathcal{H}^s$ is

$$\partial_t\,E_n = \left[\vec{\nabla}\times(\alpha\,\vec{B})\right]_n - 4\pi\,\alpha\,\vec{j}_n\,\Big|_{\mathcal{H}^s}.$$

Using

$$\left[\vec{\nabla}\times(\alpha\,\vec{B})\right]_n\Big|_{\mathcal{H}^s} = \left[\vec{\nabla}\times(4\pi\,\vec{\mathcal{J}}_H\times\vec{n})\right]_n\Big|_{\mathcal{H}^s} = -\,4\pi\,{}^{(2)}\vec{\nabla}\cdot\vec{\mathcal{J}}_H,$$

where ${}^{(2)}\vec{\nabla}$ denotes the induced covariant derivation on $\mathcal{H}^s$, we obtain

$$\partial_t\,\sigma_H + {}^{(2)}\vec{\nabla}\cdot\vec{\mathcal{J}}_H + (\alpha\,j_n)_{\mathcal{H}^s} = 0, \tag{83}$$

as an expression of charge conservation.

This completes the derivation of (69) and (70) for the Schwarzschild BH. Next, we generalize the discussion to an arbitrary static BH.

(b) Derivation for static BH

We introduce first a parametrization of the exterior metric which was used also in Israel's famous proof of the uniqueness theorem for the Schwarzschild BH.

The starting point is (3) for $\beta = 0$, i.e.,

$${}^{(4)}\boldsymbol{g} = -\alpha^2 dt^2 + \boldsymbol{g}, \quad \alpha \text{ and } \boldsymbol{g} \text{ independent of } t. \tag{84}$$

Note that $\alpha^2 = -(k|k) \geq 0$, $k = \partial_t$, and that the horizon has to be at $\alpha = 0$. We assume that the lapse function has no critical point, $d\alpha \neq 0$. The absolute space Σ is then foliated by the leaves $\{\alpha = \text{const}\}$. The function

$$\rho := (d\alpha|d\alpha)^{-\frac{1}{2}} \tag{85}$$

is then positive on Σ.

Now we introduce adapted coordinates on Σ. Consider in any point $p \in \Sigma$ the 1-dimensional subspace of $T_p(\Sigma)$ perpendicular to the tangent space of the leave $\{\alpha = \text{const}\}$ through p. This defines a 1-dimensional distribution which is, of course, involutive (integrable). The Frobenius theorem then tells us that we can introduce coordinates $\{x^i\}$ on Σ, such that x^A $(A = 2,\, 3)$ are constant along the integral curves of the distribution. For x^1 we can choose the lapse function, and thus obtain

$$\boldsymbol{g} = \rho^2 d\alpha^2 + \tilde{\boldsymbol{g}}, \quad \tilde{\boldsymbol{g}} = \tilde{g}_{AB} dx^A dx^B. \tag{86}$$

Here, ρ and $\tilde{g}_{AB}$ depend in general on all three coordinates $x^1 = \alpha$, x^A.

We also need the surface gravity κ on the horizon. A useful formula is (see, e.g., [7])

$$\kappa^2 = -\frac{1}{4}(dk|dk)\big|_H. \tag{87}$$

Now, $k = -\alpha^2 dt$, $dk = -2\alpha\, d\alpha \wedge dt$, whence

$$\kappa = \frac{1}{\rho_H}. \tag{88}$$

¿From the zeroth law of BH physics we know that $\kappa = \text{const}$. Since we want to assume a regular Killing horizon (generated by k), we conclude

$$0 < \rho_H < \infty, \qquad \rho_H = \text{const}. \tag{89}$$

Combining (84) with (86) we have outside the horizon

$${}^{(4)}\boldsymbol{g} = -N\, dt^2 + \frac{\rho^2}{4N}\, dN^2 + \tilde{\boldsymbol{g}}, \qquad N := \alpha^2. \tag{90}$$

The natural FIDO tetrad is

$$\begin{array}{rcl} \theta^0 &=& \sqrt{N}\, dt, \\ \theta^1 &=& \dfrac{\rho}{2\sqrt{N}}\, dN, \\ \theta^A && (A = 2,3): \quad \text{orthonormal 2-bein for } \tilde{\boldsymbol{g}}. \end{array} \tag{91}$$

This becomes again singular at the horizon ($N = 0$).

Now we imitate what we did for the Schwarzschild BH. We search for a basis of 1-forms which is well-defined in the neighborhood of the horizon. Guided by (72) we introduce

$$\bar{\theta}^t \equiv dt + \frac{\rho}{2N}\, dN \tag{92}$$

and rewrite (90)

$$^{(4)}\boldsymbol{g} = -N\,(\bar{\theta}^t)^2 + \rho\,\bar{\theta}^t\,dN + \tilde{\boldsymbol{g}}.$$

Thanks to (89) we conclude that

$$\left\{\bar{\theta}^t,\ dN,\ \theta^A\ (A=2,3)\right\} \tag{93}$$

remains a regular basis on the horizon. Outside the horizon we can express (91) in terms of this basis:

$$\begin{aligned}\theta^0 &= \sqrt{N}\,\bar{\theta}^t - \frac{\rho}{2\sqrt{N}}\,dN,\\ \theta^1 &= \frac{\rho}{2\sqrt{N}}\,dN.\end{aligned} \tag{94}$$

We can now proceed as in the derivation of (74), obtaining now

$$\begin{aligned}F &= \frac{\rho\,E_1}{2}\,dN\wedge\bar{\theta}^t + \sqrt{N}E_2\,\theta^2\wedge\bar{\theta}^t + \sqrt{N}E_3\,\theta^3\wedge\bar{\theta}^t + B_1\theta^2\wedge\theta^3\\ &\quad+\frac{\rho}{2\sqrt{N}}\,(E_2+B_3)\,dN\wedge\theta^2 + \frac{\rho}{2\sqrt{N}}\,(B_2-E_3)\,dN\wedge\theta^3.\end{aligned} \tag{95}$$

This implies again the limiting behavior (75) for the normal and parallel components of the electric and magnetic fields.

(c) Derivation for rotating BHs

Finally, I give a similar derivation for rotating BHs. (This was worked out in collaboration with Gerold Betschart in the course of his diploma work.)

I first need the Papapetrou parametrization of a stationary, axisymmetric BH. Since this will be treated in detail in the lectures by Markus Heusler [7], I can be brief.

Since the isometry group is $\boldsymbol{R}\times SO(2)$ we have two commuting Killing fields k and m, say, which are tangent to the orbits belonging to the group action. We assume that k and m satisfy the Frobenius integrability conditions

$$k\wedge m\wedge dk = 0, \qquad k\wedge m\wedge dm = 0. \tag{96}$$

The Frobenius theorem then guarantees that the distribution of subspaces orthogonal to k and m is (locally) integrable. I recall that (96) is implied by the field equations for vacuum spacetimes and also for certain matter models (electromagnetic fields, ideal fluids, but not for Yang-Mills fields).

In this situation, spacetime is (locally) a product manifold, $M = \Sigma\times\Gamma$, where $\Sigma = \boldsymbol{R}\times SO(2)$ and Γ is perpendicular to Σ. Thus the metric splits

$$^{(4)}\boldsymbol{g} = \boldsymbol{\sigma} + \boldsymbol{g} \tag{97}$$

such that $\boldsymbol{\sigma}$ is an invariant 2-dimensional Lorentz metric on Σ, depending, however, on $y\in\Gamma$, and the fact that $(\Gamma,\boldsymbol{g})$ is a 2-dimensional Riemannian space. In adapted coordinates

$$x^\mu:\quad x^0 = t,\quad x^1 = \varphi\ \text{for}\ \Sigma;\quad x^2,\,x^3\ \text{for}\ \Gamma, \tag{98}$$

we have

$$k = \partial_t, \qquad m = \partial_\varphi, \tag{99}$$

and

$$\boldsymbol{\sigma} = \sigma_{ab}\, dx^a dx^b, \qquad \boldsymbol{g} = g_{ij}\, dx^i dx^j, \tag{100}$$

where $a, b = 0, 1$ and $i, j = 2, 3$. The metric functions σ_{ab} and g_{ij} depend only on the coordinates x^i of Γ.

The following functions on Γ have an invariant meaning

$$-V := (k|k), \qquad W := (k|m), \qquad X := (m|m), \tag{101}$$

and we have

$$\boldsymbol{\sigma} = -V\, dt^2 + 2W\, dt d\varphi + X\, d\varphi^2. \tag{102}$$

We use also

$$\rho := \sqrt{VX + W^2} = \sqrt{-\sigma}, \qquad A := \frac{W}{X}, \tag{103}$$

in terms of which (102) takes the form

$$\boldsymbol{\sigma} = -\frac{\rho^2}{X}\, dt^2 + X(d\varphi + A\, dt)^2. \tag{104}$$

It turns out that the partial trace $R^a_{\ a}$ of the 4-dimensional Ricci tensor is proportional to ${}^{(g)}\Delta\rho$ and ρ is thus a harmonic function on Γ, whenever $R^a_{\ a}$ vanishes. This is of course the case for vacuum manifolds, but also for the Kerr-Newman solution, and in some other cases [7]. With the help of the Riemann mapping theorem one can then show that ρ is a well-defined coordinate on $(\Gamma, \boldsymbol{g})$ (ρ has no critical points). It is then possible to introduce a second coordinate z, such that

$$\boldsymbol{g} = \frac{1}{X} e^{2h} (d\rho^2 + dz^2). \tag{105}$$

In terms of t, φ, and the *Weyl coordinates* ρ, z, ${}^{(4)}\boldsymbol{g}$ assumes the *Papapetrou parametrization*

$${}^{(4)}\boldsymbol{g} = -\frac{\rho^2}{X}\, dt^2 + X\,(d\varphi + A\, dt)^2 + \frac{e^{2h}}{X}(d\rho^2 + dz^2). \tag{106}$$

We emphasize once more, that the functions X, A, and h depend only on the Weyl coordinates ρ and z.

The weak rigidity theorem tells us that on the surface on which

$$\xi = k + \Omega\, m, \qquad \Omega = -A = -W/X, \tag{107}$$

becomes null, Ω is constant and ξ is a Killing field on this surface, denoted by $H[\xi]$ in what follows. In addition, $H[\xi]$ is a stationary null surface, in particular a Cauchy horizon. (For proofs, see [7].)

Note that ξ, as a 1-form, can be expressed as

$$\xi = -\frac{\rho^2}{X}\, dt. \tag{108}$$

One also knows that X is well-behaved on the horizon

$$X = \mathcal{O}(1), \qquad X^{-1} = \mathcal{O}(1) \quad \text{on } H[\xi] \tag{109}$$

(see [8]). Since $(\xi|\xi) = -\rho^2/X$ the horizon is at $\rho = 0$ (as is well-known from the Kerr solution).

Although ξ is not a Killing field, the weak rigidity theorem implies that the surface gravity is still given by

$$\kappa^2 = -\frac{1}{4}(d\xi|d\xi)\big|_{H[\xi]}. \tag{110}$$

(A priori, the formula holds for $l := k + \Omega_H\, m$, $\Omega_H = \Omega|_{H[\xi]}$, but $d\Omega = 0$ on $H[\xi]$.) From (108) we obtain

$$d\xi = -\frac{2\rho}{X}\, d\rho \wedge dt - \rho^2\, d\left(\frac{1}{X}\right) \wedge dt.$$

Using the scalar products $(d\rho|d\rho) = X e^{-2h}$, $(dt|dt) = -X/\rho^2$, $(d\rho|dt) = 0$ gives thus, together with the zeroth law,

$$\kappa = e^{-h}\big|_{H[\xi]} = \text{const} \neq 0. \tag{111}$$

The reader should verify that this gives the correct result for the Kerr solution

$$\kappa = \frac{r_H - M}{2M r_H}. \tag{112}$$

After these preparations, our argument proceeds similarly as in (b). The natural FIDO tetrad $\{\theta^\mu\}$ for (106) is ($N := \rho^2$)

$$\theta^0 = \sqrt{\frac{N}{X}}\, dt, \quad \theta^1 = \frac{e^h}{2\sqrt{XN}}\, dN, \quad \theta^2 = \frac{e^h}{\sqrt{X}}\, dz, \quad \theta^3 = \sqrt{X}(d\varphi + A\, dt). \tag{113}$$

Clearly, θ^0 and θ^1 are again ill-defined on the horizon $N = 0$. We therefore pass over to a new basis

$$\bar\theta^t := dt + \frac{e^h}{2N}\, dN, \qquad \bar\theta^\varphi := d\varphi + \Omega\, \frac{e^h}{2N}\, dN, \tag{114}$$

together with dN and dz. In order to check whether this new basis remains valid when the horizon is approached, we express the metric (106) in terms of it. Since

$$\theta^0 = \sqrt{\frac{N}{X}}\, \bar\theta^t - \frac{e^h}{2\sqrt{XN}}\, dN, \qquad \theta^3 = \sqrt{X}(\bar\theta^\varphi + A\, \bar\theta^t), \tag{115}$$

we find readily

$$^{(4)}\boldsymbol{g} = -V(\bar\theta^t)^2 + 2W\, \bar\theta^t \bar\theta^\varphi + X(\bar\theta^\varphi)^2 + \frac{e^h}{X}\, \bar\theta^t dN + \frac{e^{2h}}{X}\, dz^2. \tag{116}$$

The determinant of the metric coefficients in this expression is $\frac{e^{4h}}{4X^2}$ and thus remains regular, thanks to (108) and (110). Since we postulate a regular horizon, it is then clear that $\{\bar\theta^t, \bar\theta^\varphi, dN, dz\}$ indeed forms a well-defined basis also on the horizon.

It is now straightforward to express the Maxwell 2-form F in terms of this basis. Instead of (96) we obtain now

$$
\begin{aligned}
F \;=\; & \left\{ E_1 \frac{e^h}{2X} + \frac{Ae^h}{2\sqrt{N}}(E_3 - B_2) \right\} dN \wedge \bar\theta^t + E_3 \sqrt{N}\, \bar\theta^\varphi \wedge \bar\theta^t \\
& \left\{ E_2 e^h \sqrt{\frac{N}{X}} + B_1 e^h A \right\} dz \wedge \bar\theta^t + \frac{e^h}{2\sqrt{N}}(B_2 - E_3)\, dN \wedge \bar\theta^\varphi \\
& + B_1 e^h\, dz \wedge \bar\theta^\varphi - \frac{e^{2h}}{2X\sqrt{N}}(E_2 + B_3)\, dz \wedge dN. \qquad (117)
\end{aligned}
$$

Since $\alpha := \sqrt{\frac{N}{X}}$ is the lapse function, we obtain once more the limiting behavior (75), and thus the basic membrane laws (69) and (70).

5 Magnetic Energy Extraction from a Black Hole

As an interesting, and possibly astrophysically important application of our basic laws (38) and (69) I show now that it is possible, in principle, to extract the rotational energy of a BH with the help of external magnetic fields. In the next section, we will work out some of the details for an ideal gedanken experiment. This will serve as a preparation for an understanding of the Blandford-Znajek process.

Our starting point is Faraday's induction law (25) in integral form, which we write down once more

$$
\mathrm{EMF}(\mathcal{C}) = -\frac{d}{dt}\int_{\mathcal{A}} \mathcal{B} + \oint_{\mathcal{C}} i_\beta\, \mathcal{B}. \qquad (118)
$$

For stationary situations this reduces to

$$
\mathrm{EMF}(\mathcal{C}) = \oint_{\mathcal{C}} i_\beta\, \mathcal{B}. \qquad (119)
$$

In Fig. 3 we consider a stationary rotating BH in an external magnetic field (like in §3). The integral in (119) along the field lines gives no contribution and far away β drops rapidly ($\sim r^{-2}$). Thus, there remains only the contribution from the horizon ($\mathcal{C}_H$) of the path $\mathcal{C}$ in Fig. 3:

$$
\mathrm{EMF} = \int_{\mathcal{C}_H} i_{\beta_H}\, \mathcal{B}, \quad \beta_H = -\Omega_H\, \partial_\varphi = -\Omega_H\, \tilde\omega_H\, e_\varphi \qquad (120)
$$

(only the normal component $\vec{B}_\perp$ contributes). I recall that $\Omega_H = a(2Mr_H)^{-1}$, $r_H = M + \sqrt{M^2 - a^2}$.

We shall show in section 6 that it is possible to construct a generator such that (with optimal impedance matching) a *maximal extraction rate* equal to

$$
\frac{1}{4}\,\frac{(\mathrm{EMF})^2}{R(\mathcal{C}_H)} \qquad (121)
$$

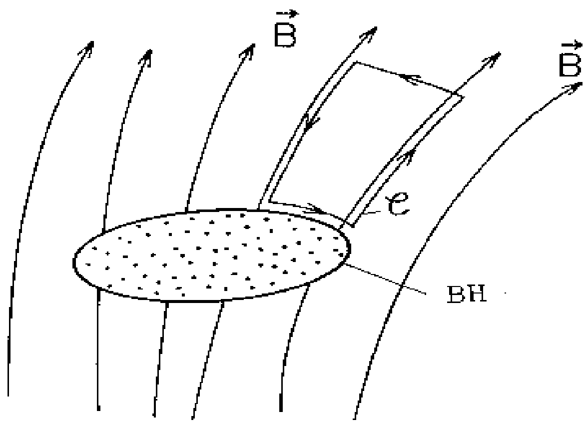

Figure 3: Arrangement for eq. (119).

becomes possible, where $R(\mathcal{C}_H)$ is the horizon ("internal") resistance

$$R(\mathcal{C}_H) = \int_{\mathcal{C}_H} R_H \, \frac{dl}{2\pi\, \tilde{\omega}}, \qquad R_H = 377 \text{ Ohm}. \tag{122}$$

(It should be known from electrodynamics, that we have to divide the surface resistivity R_H by the length of the cross section through which the current is flowing.) We shall also show that an equal amount of energy is dissipated by ohmic heating at the (stretched) horizon.

Let us work this out for the special case of an axisymmetric field: $L_{\partial_\varphi}\mathcal{B} = 0 \leftrightarrow \mathbf{d}i_{\partial_\varphi}\mathcal{B} = 0$, whence

$$i_{\partial_\varphi}\mathcal{B} = -\frac{\mathbf{d}\Psi}{2\pi}. \tag{123}$$

¿From this we conclude that $\mathcal{B}$ can be expressed in terms of two potentials Ψ and g,

$$\mathcal{B} = \underbrace{\frac{1}{2\pi}\,\mathbf{d}\Psi \wedge \mathbf{d}\varphi}_{\text{poloidal part}} + \underbrace{g * \mathbf{d}\varphi}_{\text{toroidal part}}\,, \tag{124}$$

both of which can be taken to be independent of φ. This is equivalent to the vector formulae

$$\vec{B}^{\text{pol}} = \frac{1}{2\pi\,\tilde{\omega}}\,\vec{\nabla}\Psi \times \vec{e}_\varphi, \qquad \vec{B}^{\text{tor}} = \frac{g}{\tilde{\omega}}\,\vec{e}_\varphi \tag{125}$$

(Exercise). Ψ is the magnetic flux function (see Fig. 4), because the poloidal flux inside a tube $\{\Psi = \text{const}\}$ is

$$\int \mathcal{B} = \int \mathbf{d}\Psi = \Psi, \qquad \Psi(0) = 0. \tag{126}$$

Ψ is constant along magnetic field lines, as should be clear from Fig. 4. Formally, this comes about as follows:

$$\begin{aligned} i_{\vec{B}}\,\mathbf{d}\Psi &= *(*\mathbf{d}\Psi \wedge B) = *(\mathbf{d}\Psi \wedge *B) = *(\mathbf{d}\Psi \wedge \mathcal{B}) \\ &\overset{(124)}{=} g * (\mathbf{d}\Psi \wedge *\mathbf{d}\varphi) = 0, \end{aligned}$$

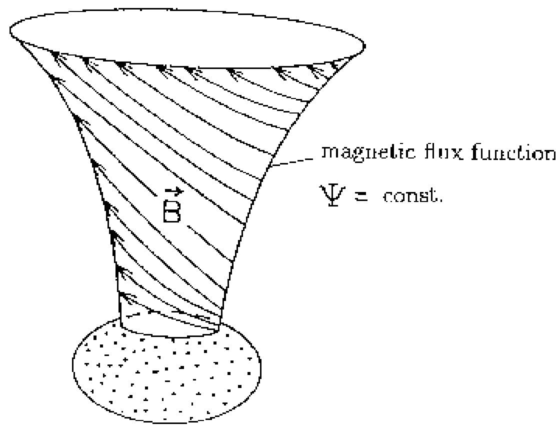

Figure 4: Axisymmetric magnetic field. The total flux inside the magnetic surface defines the flux function Ψ.

thus $< \mathbf{d}\Psi, \vec{B} >= 0$.

For the closed path $\mathcal{C}$ in Fig. 5 the EMF is by (120)

$$\text{EMF} \equiv \triangle V = \int_{\mathcal{C}_H} i_{\beta_H}\, \mathcal{B} \stackrel{(123)}{=} -\left(-\frac{\Omega_H}{2\pi}\right) \int_{\mathcal{C}_H} \mathbf{d}\Psi,$$

i.e.

$$\triangle V = \frac{\Omega_H}{2\pi}\, \triangle\Psi. \tag{127}$$

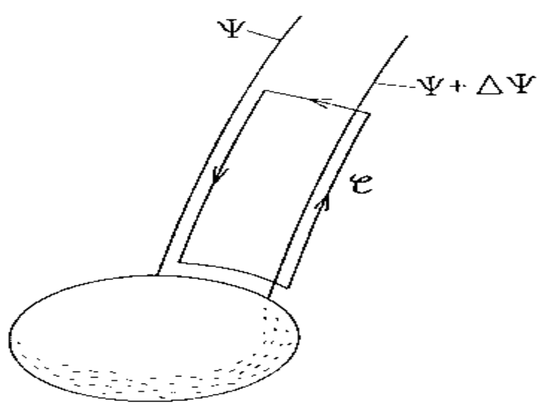

Figure 5: Integration path for the voltage in eq. (127).

The internal resistance (122) becomes

$$\triangle R_H = R_H \,\frac{\triangle l}{2\pi\, \tilde{\omega}}. \tag{128}$$

Since, on the other hand,

$$\triangle\Psi = 2\pi\, \tilde{\omega}\, B_\perp\, \triangle l, \tag{129}$$

we find, by eliminating $\triangle l$,

$$\triangle R_H = R_H \, \frac{\triangle \Psi}{4\pi^2 \, \tilde{\omega}^2 \, B_\perp}. \tag{130}$$

Inserting (127) and (130) into the expression (121) for the maximal power output gives

$$\frac{1}{4} \, \frac{(\triangle V)^2}{\triangle R_H} = \frac{\Omega_H^2}{16\,\pi} \, \tilde{\omega}^2 \, B_\perp \, \triangle \Psi. \tag{131}$$

This, as well as (127), have to be integrated from the pole to some point north of the equator (see Fig. 6). For the exact solution in §3 we know the result for the EMF, if we integrate up to the equator: From (127) and (67) we get

$$\mathrm{EMF} = \frac{1}{2\,\pi} \, \Omega_H \, 4\pi \, B_0 \, M \, (r_H - M)$$

or ($\Omega_H = a/2Mr_H$)

$$\mathrm{EMF} = a \, B_0 \, \frac{r_H - M}{r_H} \qquad \left(r_H = M + \sqrt{M^2 - a^2} \right). \tag{132}$$

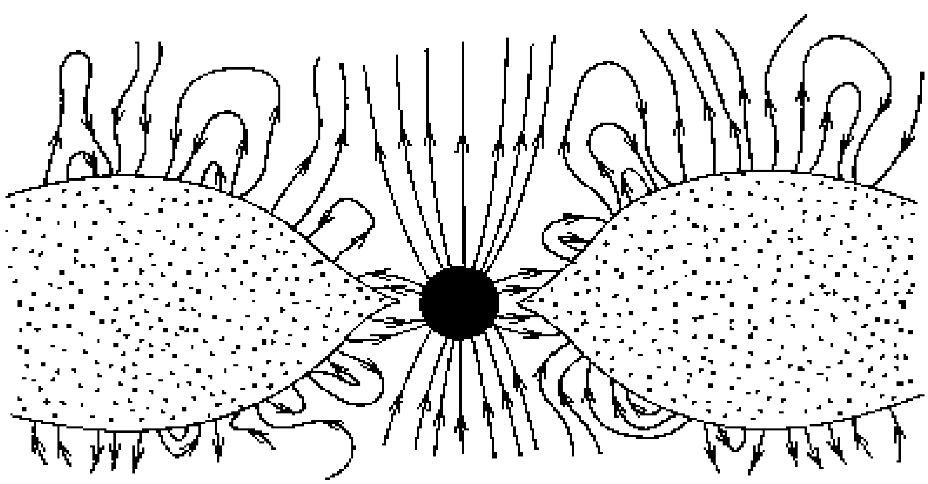

Figure 6: Plausible structure of the magnetic field close to a supermassive BH which is surrounded by an accretion disk (central engine of a quasar) (adapted from Ref. [1]).

For a general situation, like in Fig. 6, we have roughly

$$\Sigma \, \triangle \Psi = \Psi \sim B_\perp \, \pi \, r_H^2, \qquad \tilde{\omega}^2 \sim <\tilde{\omega}^2> \sim \frac{r_H^2}{2},$$

and we obtain, by (131), for the power output

$$\begin{aligned} P \;\; &\sim \;\; \frac{1}{128} \left(\frac{a}{M} \right)^2 B_\perp^2 \, r_H^2 \\ &\sim \;\; (10^{45} \, \mathrm{erg/s}) \left(\frac{a}{M} \right)^2 \left(\frac{M}{10^9 \, M_\odot} \right)^2 \left(\frac{B_\perp}{10^4 \, G} \right)^2 . \end{aligned} \tag{133}$$

The total EMF is ($V = \Sigma \triangle V$)

$$V \sim \frac{1}{2\pi}\,\Omega_H\,\Psi \sim \frac{1}{2\pi}\,\frac{a}{2Mr_H}\,B_\perp\,\pi\,r_H^2 \simeq \frac{1}{2}\left(\frac{a}{M}\right)MB_\perp \tag{134}$$

(compare this with (132)). Numerically we find

$$V \sim (10^{20}\,\mathrm{Volt})\left(\frac{a}{M}\right)\,\frac{M}{10^9\,M_\odot}\,\frac{B_\perp}{10^4\,G}. \tag{135}$$

For reasonable astrophysical parameters we obtain magnetospheric voltages $V \sim 10^{20}$ Volts and power output of the magnitude $\sim 10^{45}$ erg/s. This power is what one observes typically in active galactic nuclei, and the voltage is comparable to the highest cosmic ray energies that have been detected.

Note, however, that for a realistic astrophysical situation there is plasma outside the BH and it is, therefore, not clear, how the horizon voltage (135) is used in accelerating particles to very high energies.

Let us estimate at this point the characteristic magnetic field strength than can be expected outside a supermassive BH. A measure for this is the field strength B_E for which the energy density $B_E^2/8\pi$ is equal to the radiation energy density u_E corresponding to the Eddington luminosity

$$L_E = \frac{4\pi M_H m_p c}{\sigma_T} = 1.3\times 10^{38}\,(\mathrm{erg/s})\,\frac{M}{M_\odot}. \tag{136}$$

The relation between L_E and u_E is

$$L_E = 4\pi\,r_g^2\,\frac{c}{4}\,u_E = \pi\,r_g^2\,c\,u_E \qquad \left(r_g = \frac{G\,M}{c^2}\right). \tag{137}$$

Thus

$$\frac{1}{8\,\pi}\,B_E^2 = \frac{4\,m_p c^2}{\sigma_T\,r_g}, \tag{138}$$

giving ($M_{H,8} \equiv M_H/10^8\,M_\odot$)

$$B_E = 1.2\times 10^5\,M_{H,8}^{-1/2}\ \mathrm{Gauss}. \tag{139}$$

For a BH with mass $\sim 10^9\ M_\odot$ inside an accretion disk acting as a dynamo, a characteristic field of about 1 Tesla (10^4 Gauss) is thus quite reasonable.

6 Rotating BH as a Current Generator

Before we come to realistic possibilities of energy extraction, we analyze in detail an idealized arrangement, sketched in Fig. 7.

We compute first the EMF around a closed path consisting of the following parts: We start from the equator of the (stretched) horizon along a perfectly conducting disk, which is supposed to rotate differentially in such a manner, that all its pieces are at rest relative to the FIDOs (they have thus zero angular momentum). From the boundary of the disk, the path continuous along a wire and through a resistive load (R_L) to the top of a conical conductor. Then we move down to its tip at $\vartheta = \vartheta_0$

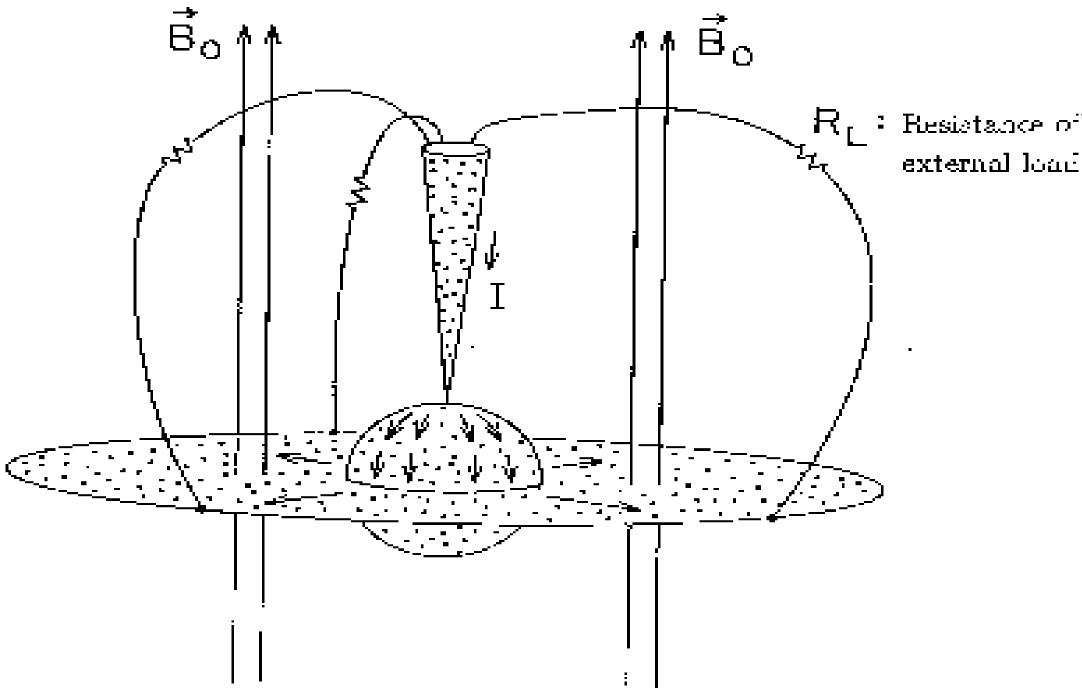

Figure 7: An idealized current generator (adapted from [1]).

close to the north pole of the membrane, and finally down the horizon in the poloidal direction to the starting point. (The conical surface is chosen instead of an infinitely thin wire in order to avoid divergent integrals; it replaces a wire of finite thickness.)

The ideally conducting disk gives no contribution to the EMF. Formally this comes about as follows: The 4-velocity field of the disk u (whose parts move with the FIDOs) is e_0 (see (5)) and thus $-i_u F = E = 0$, for an ideal conductor. Using (7), this implies $\mathcal{E} = 0$.

The contribution V_L to the line integral $\oint \alpha \, \mathcal{E}$ from the load resistance far from the BH is what we all know from electrodynamics

$$V_L = I \, R_L, \tag{140}$$

where I is the current in the wire.

There remains the contribution V_H to the EMF along the stretched horizon

$$V_H = \int_{\mathcal{C}_H} \alpha \, \mathcal{E} \overset{\text{(Ohm)}}{=} \int_{\mathcal{C}_H} R_H * \mathcal{J}_H. \tag{141}$$

The current flows along the northern hemisphere of the horizon as a surface current in the *poloidal* direction, with surface current density

$$\vec{\mathcal{J}}_H = \frac{I}{2\pi \, \tilde{\omega}_H} \, \vec{e}_\vartheta. \tag{142}$$

In order to prove that the surface current $\vec{\mathcal{J}}_H$ has no toroidal component, we apply (119) to a toroidal path $\{\vartheta = \text{const}\}$ on the stretched horizon. Along this $i_\beta \, \mathcal{B} = -\mathbf{d}\Psi/2\pi = 0$, and thus

$$\oint \alpha \, \mathcal{E} = 0 \quad \Longrightarrow \quad i_{\partial_\varphi} \, \mathcal{E}_H = 0 : \; \vec{E}_H \cdot \vec{e}_\varphi = 0.$$

Ohm's law in (69) then implies $\vec{\mathcal{J}}_H \cdot \vec{e}_\varphi = 0$.

Using (142) in (141) gives

$$V_H = I \int_{\vartheta_0}^{\frac{\pi}{2}} R_H \, \frac{\rho_H}{2\pi\, \tilde{\omega}_H} \, d\vartheta$$

or

$$V_H = I \, R_{HT}, \tag{143}$$

where

$$R_{HT} = \int_{\vartheta_0}^{\frac{\pi}{2}} R_H \, \frac{\rho_H}{2\pi\, \tilde{\omega}_H} \, d\vartheta$$

is the total resistance of the horizon (see (122)). The EMF around our closed path is thus

$$V = V_H + V_L = I(R_{HT} + R_L). \tag{144}$$

This voltage is also equal to the line integral on the right hand side in (119),

$$V = \oint_{\mathcal{C}} i_\beta \, \mathcal{B}. \tag{145}$$

This receives only contributions from the horizon and the disk (the contribution of the latter was overlooked in Ref. [1]). Using $\beta = -\omega \, \partial_\varphi$, $\mathcal{B}_\perp = B_r \, \vartheta^\vartheta \wedge \vartheta^\varphi$, $\vartheta^\vartheta = \rho \, d\vartheta$, $\vartheta^\varphi = \tilde{\omega} \, d\varphi$, the horizon gives

$$\int_{\mathcal{C}_H} i_\beta \, \mathcal{B} = \Omega_H \int_{\vartheta_0}^{\frac{\pi}{2}} B_\perp \, \tilde{\omega}_H \, \rho_H \, d\vartheta. \tag{146}$$

A similar contribution is obtained along the disk ($\mathcal{C}_D$) and the total voltage is given by

$$V = \Omega_H \int_{\vartheta_0}^{\frac{\pi}{2}} B_\perp \, \tilde{\omega}_H \, \rho_H \, d\vartheta - \int_{r_H}^{r_D} B_\perp \, \omega \, \tilde{\omega} \, \frac{\rho}{\sqrt{\Delta}} \, dr \tag{147}$$

(r_D is the edge of the disk). This "battery" voltage[1], and the total horizon and load resistances R_{HT} and R_L determine the current I according to (144).

If the magnetic field is axisymmetric, we can use (123) to write the integrand in (145) as follows

$$i_\beta \, \mathcal{B} = \frac{\omega}{2\,\pi} \, \mathbf{d}\Psi. \tag{148}$$

The voltage is then

$$V = \frac{\Omega_H}{2\,\pi} \, \Psi(\mathrm{eq}) \left[1 + \int_{horizon}^{edge} \left(\frac{\omega}{\Omega_H} \right) \mathbf{d} \left(\frac{\Psi}{\Psi(\mathrm{eq})} \right) \right], \tag{149}$$

[1]The part of the integral (145) from the disk is independent of the connecting path, because the induction law gives $\mathbf{d}(\alpha \, \mathcal{E} - i_\beta \, \mathcal{B}) = 0$ and thus $\mathbf{d}\, (i_\beta \, \mathcal{B}) = 0$ inside the disk.

where $\Psi(\text{eq})$ denotes the value of the flux function at the equator of the horizon. The two pieces in the square bracket are comparable in magnitude.

We shall see in the next section in detail how the power, dissipated as ohmic losses

$$P_L = I^2 R_L \tag{150}$$

in the load, is extracted from the hole, but this is clearly at the cost of the mass of the BH:

$$I^2 R_L = -\dot{M}. \tag{151}$$

Let us note that

$$P_L = V^2 \frac{R_L}{(R_{HT} + R_L)^2}. \tag{152}$$

This becomes maximal for

$$R_{HT} = R_L \qquad \text{(impedance matching)} \tag{153}$$

with

$$P_L^{max} = \frac{V^2}{4\, R_{HT}}. \tag{154}$$

This maximal extraction rate was stated in (121).

Clearly, we can also reverse our gedanken experiment. By applying a voltage, we can use the BH as the rotator of an electric motor (see Fig. 8). You see that some of the physics of BHs is indeed very similar to that of ordinary electric generators and electric motors.

7 Conservation Laws, Increase of Entropy of a BH

The general results which will be obtained in this section will enable us to develop a more profound analysis of the idealized current generator discussed above.

Contraction of the energy-momentum tensor $T^{\mu\nu}$ with the two Killing fields $\partial_t,\ \partial_\varphi$ gives two conserved vector fields, which express the conservation laws of energy and angular momentum in the z-direction. We want to formulate these in the 3+1 splitted form.

To this end, we consider as a preparation a 4-dimensional equation of the type

$$\nabla \cdot J = Q \tag{155}$$

for a vector field J with source term Q. The Hodge dual of the 1-form $J^\flat$ has the same decomposition as $\mathcal{S}$ in (22):

$$*J^\flat = \rho + (i_\beta\, \rho - \alpha\, \mathcal{J}) \wedge dt. \tag{156}$$

If $J = J^\mu e_\mu$, then ρ is the 3-form belonging to J^0, and $\mathcal{J}$ is the 2-form corresponding to $\vec{j} = J^k e_k$. Eq. (155) is equivalent to $d\left(*J^\flat\right) = Q\,\mathrm{vol}_4$ or

$$dt \wedge [(\partial_t - L_\beta)\rho + \mathbf{d}(\alpha\, \mathcal{J})] = Q\,\mathrm{vol}_4. \tag{157}$$

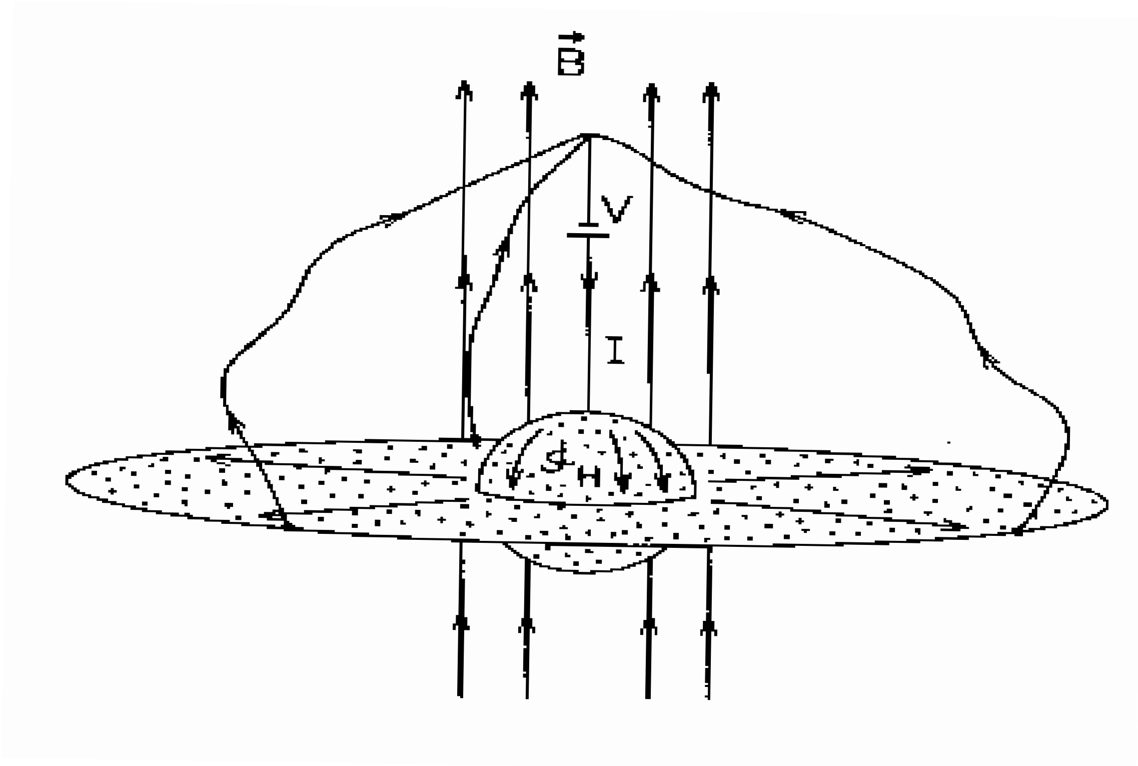

Figure 8: Black hole playing the role of the rotator of an electric motor.

Since $dt = \theta^0/\alpha$, the 3+1 split is

$$(\partial_t - L_\beta)\,\rho + \mathbf{d}(\alpha\,\mathcal{J}) = \alpha\,Q\,\mathrm{vol}_3. \tag{158}$$

If $\operatorname{div}\vec{\beta} = 0$ (as for the Kerr solution), this is equivalent to ($\rho =: \hat{\rho}\,\mathrm{vol}_3$)

$$(\partial_t - L_{\vec{\beta}})\,\hat{\rho} + \vec{\nabla}\cdot\left(\alpha\,\vec{j}\right) = \alpha\,Q. \tag{159}$$

The integral form

$$\frac{d}{dt}\int_{\mathcal{V}}\rho = -\int_{\partial\mathcal{V}}(\alpha\,\mathcal{J} - i_\beta\,\rho) + \int_{\mathcal{V}}\alpha\,Q\,\mathrm{vol}_3 \tag{160}$$

generalizes the conservation law (28). We also write (160) in vector analytic form

$$\frac{d}{dt}\int_{\mathcal{V}}\hat{\rho}\,dV = -\int_{\partial\mathcal{V}}(\alpha\,\vec{j} - \hat{\rho}\,\vec{\beta})\cdot d\vec{A} + \int_{\mathcal{V}}\alpha\,Q\,dV. \tag{161}$$

Let us now apply this to the vector fields $k_\mu T^{\mu\nu}$ und $m_\mu T^{\mu\nu}$ ($k = \partial_t,\ m = \partial_\varphi$). The corresponding $\hat{\rho}$ and $\vec{j}$ are denoted as follows

$$\begin{aligned} -\,k\cdot T &\longleftrightarrow (\varepsilon_{E_\infty},\, \vec{S}_{E_\infty}),\\ m\cdot T &\longleftrightarrow (\varepsilon_{L_z},\, \vec{S}_{L_z}). \end{aligned} \tag{162}$$

Eq. (159) gives

$$(\partial_t - L_{\vec{\beta}})\,\varepsilon_{E_\infty} + \vec{\nabla}\cdot(\alpha\,\vec{S}_{E_\infty}) = 0 \qquad \text{(Poynting theorem)}, \tag{163}$$

$$(\partial_t - L_{\vec{\beta}})\,\varepsilon_{L_z} + \vec{\nabla}\cdot(\alpha\,\vec{S}_{L_z}) = 0 \quad \text{(angular momentum conservation)}. \tag{164}$$

The corresponding integral formulas are

$$\frac{d}{dt}\int_{\mathcal{V}} \varepsilon_{E_\infty}\, dV \;=\; -\int_{\partial\mathcal{V}} \left(\alpha\,\vec{S}_{E_\infty} - \varepsilon_{E_\infty}\,\vec{\beta}\right)\cdot d\vec{A}, \tag{165}$$

$$\frac{d}{dt}\int_{\mathcal{V}} \varepsilon_{L_z}\, dV \;=\; -\int_{\partial\mathcal{V}} \left(\alpha\,\vec{S}_{L_z} - \varepsilon_{L_z}\,\vec{\beta}\right)\cdot d\vec{A}. \tag{166}$$

Now we make use of (5), i.e., $\partial_t = \alpha\, e_0 - \omega\,\partial_\varphi = \alpha\, e_0 - \omega\,\tilde{\omega}\,\vec{e}_\varphi$, giving us

$$-k\cdot T = -\alpha\, e_0\cdot T + \omega\, m\cdot T.$$

Here we use the following FIDO decomposition of T:

$$T = \varepsilon\, e_0\otimes e_0 + e_0\otimes\vec{S} + \vec{S}\otimes e_0 + \overleftrightarrow{\mathbf{T}}, \tag{167}$$

and obtain the relations

$$\varepsilon_{E_\infty} \;=\; \alpha\,\varepsilon + \omega\,\varepsilon_{L_z}, \qquad \text{(energy density “at infinity”)}, \tag{168}$$

$$\vec{S}_{E_\infty} \;=\; \alpha\,\vec{S} + \omega\,\vec{S}_{L_z}, \quad \text{(energy current density “at infinity”)}. \tag{169}$$

Application to a Kerr BH

The global conservation laws (165) and (166) are now applied to a Kerr BH. Its mass plays the role of the “energy at infinity” and thus, the energy conservation (165) with (169), gives

$$\frac{dM}{dt} = -\int_{\mathcal{H}^s} \alpha_H\,\vec{S}_{E_\infty}\cdot\vec{n}\, dA = -\int_{\mathcal{H}^s}\left(\alpha_H^2\,\vec{S} + \alpha_H\,\Omega_H\,\vec{S}_{L_z}\right)\cdot\vec{n}\, dA. \tag{170}$$

Similarly, the change of the angular momentum of the BH is by (166)

$$\frac{dJ}{dt} = -\int_{\mathcal{H}^s} \alpha_H\,\vec{S}_{L_z}\cdot\vec{n}\, dA; \qquad \vec{S}_{L_z} = \vec{\partial}_\varphi\cdot \overleftrightarrow{\mathbf{T}}\;. \tag{171}$$

Now, we consider the entropy increase of the BH. The first law [7] tells us that

$$T_H\,\frac{dS_H}{dt} = \frac{dM}{dt} - \Omega_H\,\frac{dJ}{dt}. \tag{172}$$

On the right hand side we insert (170) and (171), and use also (169), giving us the generally valid formula

$$T_H\,\frac{dS_H}{dt} = -\int_{\mathcal{H}^s} \alpha_H^2\,\vec{S}\cdot\vec{n}\, dA. \tag{173}$$

Until now we have not specified the matter content. For electrodynamics we have $\vec{S} = \frac{1}{4\pi}\,\vec{E}\times\vec{B}$, and thus at the horizon $\alpha_H^2\,\vec{S} = \frac{1}{4\pi}\,\vec{E}_H\times\vec{B}_H$ (see (78)). In this case we obtain with the laws of Ampère and Ohm

$$T_H\,\frac{dS_H}{dt} = -\frac{1}{4\pi}\int_{\mathcal{H}^s} (\vec{E}_H\times\vec{B}_H)\cdot\vec{n}\, dA = \int_{\mathcal{H}^s} \vec{\mathcal{J}}_H\cdot\vec{E}_H\, dA = \int_{\mathcal{H}^s} R_H\,\vec{\mathcal{J}}_H^2\, dA. \tag{174}$$

All of these familiarly looking expressions for the rate of entropy increase are important and useful.

Let us also evaluate (171) in a similar manner. First, we have

$$\vec{S}_{L_z} = \frac{1}{4\pi}\,\vec{\partial}_\varphi \cdot \left[-\left(\vec{E}\otimes\vec{E} + \vec{B}\otimes\vec{B}\right) + \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)\overleftrightarrow{\mathbf{g}}\right]. \tag{175}$$

Clearly, only the first term contributes to the integrand in (171). Using this time the laws of Gauss and Ampère, we find

$$\frac{dJ}{dt} = \int_{\mathcal{H}^s} \left(\sigma_H\,\vec{E}_H + \vec{\mathcal{J}}_H \times \vec{B}_n\right)\cdot\vec{\partial}_\varphi\,dA. \tag{176}$$

Note that the first term is absent if $\vec{E}_H$ has no toroidal component.

Finally, we use (172), together with (174) and (176), to obtain the following formula for the change of the mass of the BH

$$\frac{dM}{dt} = \int_{\mathcal{H}^s} \left[\vec{\mathcal{J}}_H\cdot\vec{E}_H - \vec{\beta}_H\cdot\left(\sigma_H\,\vec{E}_H + \vec{\mathcal{J}}_H\times\vec{B}_n\right)\right] dA. \tag{177}$$

Application to the idealized current generator

It is instructive to use these general results for a more detailed analysis of the current generator, discussed in the last section.

We begin by computing the ohmic heating rate of the current flowing through the northern hemisphere (n.H.) of the BH:

$$\begin{aligned}\int_{n.H.} \vec{E}_H\cdot\vec{\mathcal{J}}_H\,dA \quad &= \quad \int_{\vartheta_0}^{\frac{\pi}{2}} E_{H\hat{\vartheta}}\,\mathcal{J}_{H\hat{\vartheta}}\,2\pi\,\tilde{\omega}\,\rho_H\,d\vartheta \\ &\overset{(142)}{=} \quad I\int_{\vartheta_0}^{\frac{\pi}{2}} E_{H\hat{\vartheta}}\,\rho_H\,d\vartheta = I\,V_H \overset{(143)}{=} I^2\,R_{HT}.\end{aligned} \tag{178}$$

According to the general result (174) this rate is equal to $T_H\,dS_H/dt$. Thus,

$$T_H\,\frac{dS_H}{dt} = I^2\,R_{HT}. \tag{179}$$

In order to trace the details of the energy flow, we apply the generalized Poynting theorem (165). For a stationary situation this reduces to

$$\int_{\partial\mathcal{V}} \left(\alpha\,\vec{S}_{E_\infty} - \varepsilon_{E_\infty}\,\vec{\beta}\right)\cdot d\vec{A} = 0. \tag{180}$$

We choose the volume $\mathcal{V}$ such that the boundary ∂V consists of a horizon part $\mathcal{A}_H$ ($\mathcal{H}^s$ minus the inner edge of the disk), the disk $\mathcal{A}_D$, and a surface $\mathcal{A}_L$ enclosing the load's resistor (see Fig. 9). The second term in (180) does not contribute for $\mathcal{A}_H$ and $\mathcal{A}_D$, and can be ignored for the load, since this is assumed to be located far from the horizon. Using also the relation (169) we have then

$$\int_{\partial\mathcal{V}} \alpha\,\vec{S}_{E_\infty}\cdot d\vec{A} = \int_{\partial\mathcal{V}} (\alpha^2\vec{S} + \alpha\,\omega\,\vec{S}_{L_z})\cdot d\vec{A} = 0. \tag{181}$$

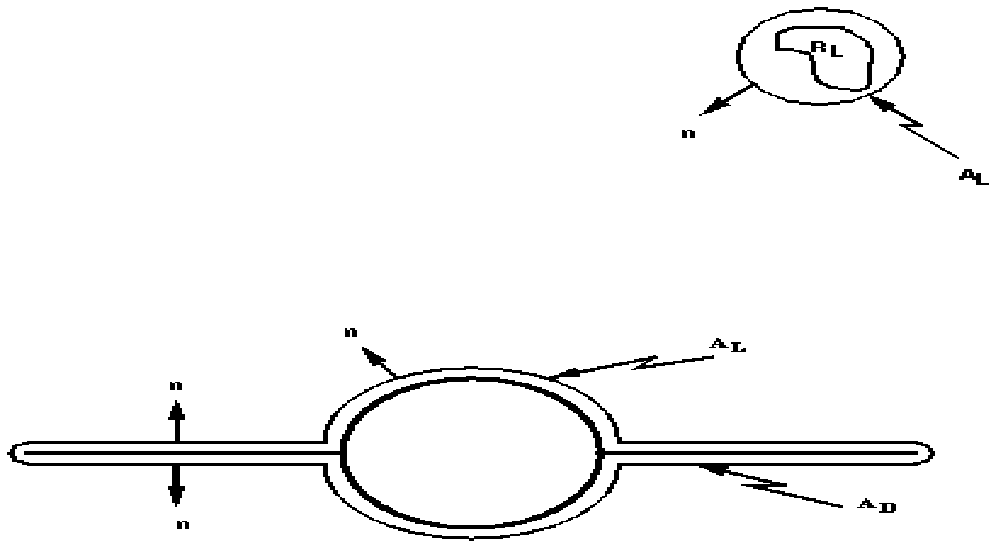

Figure 9: The boundary $\partial V = \mathcal{A}_H \cup \mathcal{A}_D \cup \mathcal{A}_L$ for (181)

The contribution from the horizon to the first term on the right is

$$-\int_{\mathcal{A}_H} \alpha^2 \vec{S} \cdot \vec{n}\, dA \overset{(173)}{=} T_H \frac{dS_H}{dt} \overset{(179)}{=} I^2 R_{HT}, \tag{182}$$

and the second term gives

$$-\Omega_H \int_{\mathcal{A}_H} \alpha_H\, \vec{S}_{L_z} \cdot \vec{n}\, dA. \tag{183}$$

At first sight one might think that this is just $\Omega_H \frac{dJ}{dt}$ (see (171)). This is, however, not correct, because there is an additional inflow of angular momentum through the disk. Physically, this is clear: The external magnetic field exerts – through the Lorentz force on the surface current $\vec{\mathcal{J}}_D$ of the disk – a torque on the disk, and thus on the BH to which the disk is locked. As in the derivation of (176) we find for (183)

$$\begin{aligned}
-\Omega_H \int_{\mathcal{A}_H} \alpha_H\, \vec{S}_{L_z} \cdot \vec{n}\, dA \quad &= \quad \Omega_H \int_{\mathcal{A}_H} \tilde{\omega}_H (\vec{\mathcal{J}}_H \times \vec{B}_n) \cdot \vec{e}_\varphi\, dA \\
= -\Omega_H \int_{\vartheta_0}^{\frac{\pi}{2}} \mathcal{J}_{H\hat{\vartheta}}\, B_\perp\, \tilde{\omega}_H\, 2\pi\, \tilde{\omega}_H\, \rho_H\, d\vartheta \quad &\overset{(142)}{=} \quad -I\, \Omega_H \int_{\vartheta_0}^{\frac{\pi}{2}} B_\perp\, \tilde{\omega}_H\, \rho_H\, d\vartheta,
\end{aligned}$$

where use has been made of the fact, that $\vec{E}_H$ has no toroidal component. Together with (146), we obtain for this horizon contribution

$$-\Omega_H \int_{\mathcal{A}_H} \alpha_H\, \vec{S}_{L_z} \cdot \vec{n}\, dA = -I \int_{\mathcal{C}_H} i_\beta\, \mathcal{B}. \tag{184}$$

Note that the integral on the right is not the total voltage (145). However, we shall see shortly that the disk contributes the remaining part of $-I\,V$.

Only the last term in (181) gets contributions from the disk (since $\vec{E} = 0$ in the disk). Using the expression (175) for $\vec{S}_{L_z}$, this becomes

$$-\int_{\mathcal{A}_D} \alpha\, \omega\, \vec{S}_{L_z} \cdot \vec{n}\, dA = \int_{\mathcal{C}_D} \alpha\, \omega\, \frac{B_n}{4\pi}\, \vec{\partial}_\varphi \cdot \vec{B}\, 2\pi \tilde{\omega}\, dl. \tag{185}$$

Between the parallel component $\vec{B}_\parallel$ and the surface current $\vec{\mathcal{J}}_D$ of the disk we have Ampère's relation of ordinary electrodynamics:

$$\alpha\, \vec{B}_\parallel = 4\pi\, \vec{\mathcal{J}}_D \times \vec{n} = 4\pi\, \frac{I}{2\pi\, \tilde{\omega}}\, \vec{e}_r \times \vec{n}. \tag{186}$$

Making use of this, (185) becomes

$$-\int_{\mathcal{A}_D} \alpha\, \omega\, \vec{S}_{L_z} \cdot \vec{n}\, dA = -I \int_{\mathcal{C}_D} \vec{\beta} \cdot \left(\vec{e}_r \times \vec{B}_n\right)\, dl = I \int_{\mathcal{C}_D} \vec{\beta} \cdot \left(\vec{\beta} \times \vec{B}_n\right) \cdot d\vec{l}$$

or

$$-\int_{\mathcal{A}_D} \alpha\, \omega\, \vec{S}_{L_z} \cdot \vec{n}\, dA = -\int_{\mathcal{C}_D} i_\beta\, \mathcal{B}, \tag{187}$$

as already announced.

Finally, the contribution to (181) is what we are used to:

$$\int_{\mathcal{A}_L} \alpha\, \vec{S}_{E_\infty} \cdot d\vec{A} = \int_{\mathcal{A}_L} \frac{1}{4\pi}\, (\vec{E} \times \vec{B})\, d\vec{A} = I^2 R_L. \tag{188}$$

All together, the Poynting theorem (181) gives

$$\underbrace{I^2 R_{TH} - I \int_{\mathcal{C}_H} i_\beta\, \mathcal{B}}_{\text{horizon}} \underbrace{- I \int_{\mathcal{C}_D} i_\beta\, \mathcal{B}}_{\text{disk}} = \underbrace{-I^2 R_L}_{\text{load}}. \tag{189}$$

¿From the derivation it is clear how to interpret this result. First, we note that the left hand side is the energy (measured at infinity) which flows per unit time into the BH, and thus is equal to dM/dt. According to (179) the first term is $T_H\, dS_H/dt$ (ohmic dissipation). The second term in (189) is that part of $\Omega_H\, dJ/dt$ which is due to the torque acting on the surface current density $\vec{\mathcal{J}}_H$ (see (171) and (176)). The third term resulted from the disk and gives an additional contribution to the change of the rotational energy, which flows through the disk into the BH. For clarification we note that for a closed surface $\mathcal{A}$ surrounding the disk we have

$$\oint_{\mathcal{A}} \alpha\, \vec{S}_{E_\infty} \cdot d\vec{A} = 0 = \oint_{\mathcal{A}} \alpha\, \omega\, \vec{S}_{L_z} \cdot d\vec{A},$$

since $\vec{S}$ vanishes in the disk. Therefore, the rotational energy which flows through the inner edge of the disk into the BH is (see (171))

$$\Omega_H \int_{\leftarrow|} \alpha\, \vec{S}_{L_z} \cdot d\vec{A} = -\int_{\mathcal{A}_D} \alpha\, \omega\, \vec{S}_{L_z} \cdot \vec{n}\, dA \overset{(187)}{=} -\int_{\mathcal{C}_D} i_\beta\, \mathcal{B}.$$

The total change of the rotational energy of the BH thus is

$$\begin{aligned} \Omega_H \frac{dJ}{dt} &= -I \oint_{\mathcal{C}} i_\beta\, \mathcal{B} \overset{(145)}{=} -I\, V \\ &\overset{(144)}{=} -I^2 (R_{HT} + R_L). \end{aligned} \tag{190}$$

The presence of the disk made the discussion a bit complicated, but it is nice to see how the various pieces combine. Schematically, we have

$$\begin{array}{ccccc} \frac{dM}{dt} & = & T_H\,\frac{dS_H}{dt} & + & \Omega_H\,\frac{dJ}{dt} \\ \| & & \| & & \| \\ -I^2 R_L & & I^2 R_{HT} & & -I^2(R_{HT}+R_L). \end{array} \tag{191}$$

The energy (at ∞) dM/dt flows – partly through the disk – down the hole and is, of course, negative because the spin down overcomes the ohmic dissipation. This part of the energy balance results, therefore, in an outflowing energy which is carried without loss by the electromagnetic fields to the load resistors, where it is dissipated at the rate $I^2 R_L$. Netto, we obtain, of course,

$$-\frac{dM}{dt} = I^2 R_L. \tag{192}$$

8 Blandford-Znajek Process

Let us return to Fig. 6, in which a plausible magnetic field structure around a supermassive BH in the center of an active galaxy is sketched. Rotation and turbulence in an accretion disk can generate magnetic fields of the order 1 tesla, as we estimated at the end of section 5.

Close to the BH one expects a force-free electron-positron plasma, which comes about as follows. Imagine first that there are no charged particles in the neighborhood of the hole. In this case unipolar induction generates a quadrupole-like electric field similar to that we found in section 3 (see Fig. 2). Close to the BH the magnitude of this electric field is $E \sim B\,a/M \sim 3\times 10^6$ (Volt/cm) (a/M). Between the horizon and a few gravitational radii along the magnetic field lines this gives rise to a voltage $V \sim E\,r_H \sim B\,a \sim 10^{20}$ Volt for a BH with mass $M \sim 10^9\,M_\odot$ (see section 5). In this enormous potential stray electrons from the disk or interstellar space will be accelerated along magnetic field lines to ultrarelativistic energies. Inverse Compton scattering with soft photons from the accretion disk leads to γ-quanta which in turn annihilate with soft photons from the disk into electron-positron pairs. These will again be accelerated and by repetition an electron-positron plasma is generated which can become dense enough to annihilate the component of $\vec{E}$ along $\vec{B}$. The electric field is then nearly orthogonal to $\vec{B}$, up to a sufficiently large component which produces occasional electron-positron sparks in order to fill the magnetosphere with plasma. (Similar processes are important in the magnetospheres of pulsars. It is likely that all active pulsars have electron-positron winds.)

Fields with $\vec{E}\cdot\vec{B} = 0$ are called *degenerate*. We assume in what follows that in the neighborhood of the BH, where the $\vec{B}$-field is strong, the electromagnetic field is *force-free*, which means that the ideal MHD condition

$$\rho_e\,\vec{E} + \vec{j}\times\vec{B} = 0 \tag{193}$$

is satisfied. Clearly, in a force-free plasma $\vec{E}$ is perpendicular to $\vec{B}$. Furthermore, $\vec{E}$ has no toroidal component for an axisymmetric stationary situation. This is an immediate consequence of the induction law: Applying (25) for a stationary and axisymmetric configuration to the path $\mathcal{C}$ in Fig 10, we obtain

$$\oint_{\mathcal{C}} \alpha\,\mathcal{E} = \oint_{\mathcal{C}} i_\beta\,\mathcal{B} \overset{(123)}{=} -\frac{1}{2\pi}\oint_{\mathcal{C}} \mathbf{d}\Psi = 0 \Longrightarrow \vec{E}^{\rm tor} = 0. \tag{194}$$

Similarly, Ampère's law in integral form (27) reduces to

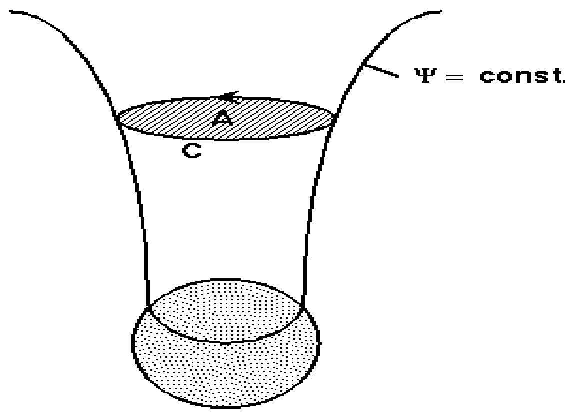

Figure 10: Path for (194) and (195).

$$\oint_{\mathcal{C}} \alpha\, \mathcal{H} = 4\,\pi \oint_{\mathcal{A}} \alpha\, \mathcal{J} = 4\,\pi\, I, \tag{195}$$

where I is the total upward current through a surface $\mathcal{A}$ bounded by $\mathcal{C}$. This gives for the toroidal component of the magnetic field

$$\vec{B}^{\mathrm{tor}} = \frac{2\,I}{\alpha\,\tilde{\omega}}\,\vec{e}_{\varphi}. \tag{196}$$

Using this in (124) gives

$$\mathcal{B} = \underbrace{\frac{1}{2\,\pi}\,\mathbf{d}\Psi \wedge \mathbf{d}\varphi}_{\text{poloidal}} + \underbrace{\frac{2\,I}{\alpha}\; *\,\mathbf{d}\varphi}_{\text{toroidal}}\,. \tag{197}$$

For later use, we also note the following: The continuity equation (24) reduces to

$$\mathbf{d}\,(\alpha\,\mathcal{J}) = 0. \tag{198}$$

Moreover, $L_{\partial_\varphi}\,(\alpha\,\mathcal{J}) = 0$, we have $\mathbf{d}\,(i_{\partial_\varphi}\,\alpha\,\mathcal{J}) = 0$, thus

$$i_{\partial_\varphi}\,(\alpha\,\mathcal{J}) = \frac{1}{2\,\pi}\,\mathbf{d}I \tag{199}$$

or

$$\alpha\,\mathcal{J}^{\mathrm{pol}} = \frac{1}{2\,\pi}\,\mathbf{d}I \wedge \mathbf{d}\varphi. \tag{200}$$

Clearly, the potential I is the current in (195).

Because $\mathcal{E}$ is poloidal, we can represent the electric field as follows

$$\mathcal{E} = i_{\vec{v}_F}\,\mathcal{B} \qquad (\vec{E} = -\vec{v}_F \times \vec{B}), \tag{201}$$

where $\vec{v}_F$ is toroidal. Let us set

$$\vec{v}_F =: \frac{1}{\alpha}\,(\Omega_F - \omega)\,\tilde{\omega}\,\vec{e}_{\varphi}. \tag{202}$$

For the interpretation of Ω_F note the following: For an observer, rotating with angular velocity Ω, the 4-velocity is $u = u^t\,(\partial_t + \Omega\,\partial_\varphi)$. On the other hand, $u = \gamma\,(e_0 + \vec{v})$, where $\vec{v}$ is the 3-velocity relative to a FIDO. Using also $\partial_t = \alpha\,e_0 + \vec{\beta}$ we get $\Omega\,\vec{\partial}_\varphi = \alpha\,\vec{v} - \vec{\beta}$ or

$$\vec{v} = \frac{1}{\alpha}\,(\Omega - \omega)\,\vec{\partial}_\varphi = \frac{1}{\alpha}\,(\Omega - \omega)\,\tilde{\omega}\,\vec{e}_\varphi. \tag{203}$$

This has the same form as (202) and, therefore, Ω_F is the angular velocity of the magnetic field lines.

Next, we show that Ω_F is only a function of Ψ. The induction law gives

$$\begin{aligned} \mathbf{d}(\alpha\,\mathcal{E}) &= L_\beta\,\mathcal{B} = -L_{\omega\,\partial_\varphi}\,\mathcal{B} = -\omega\,L_{\partial_\varphi}\,\mathcal{B} - \mathbf{d}\omega \wedge i_{\partial_\varphi}\,\mathcal{B} \\ &= \frac{1}{2\,\pi}\,\mathbf{d}\omega \wedge \mathbf{d}\Psi. \end{aligned}$$

On the other hand, (201) and (202) give

$$\begin{aligned} \mathbf{d}(\alpha\,\mathcal{E}) &= \mathbf{d}(\alpha\,i_{\vec{v}_F}\,\mathcal{B}) = \mathbf{d}\left((\Omega_F - \omega)\,i_{\partial_\varphi}\,\mathcal{B}\right) = -\mathbf{d}\left(\frac{\Omega_F - \omega}{2\,\pi}\,\mathbf{d}\Psi\right) \\ &= -\mathbf{d}\left(\frac{\Omega_F - \omega}{2\,\pi}\right) \wedge \mathbf{d}\Psi. \end{aligned}$$

By comparison, we get $\mathbf{d}\Omega_F \wedge \mathbf{d}\Psi = 0 \Longrightarrow \Omega_F = \Omega_F(\Psi)$. The calculation above also shows

$$\alpha\,\mathcal{E} = -\frac{\Omega_F - \omega}{2\,\pi}\,\mathbf{d}\Psi, \tag{204}$$

i.e., $\vec{E}$ is *perpendicular* to the surfaces $\{\Psi = \text{const}\}$.

Specializing to the horizon gives

$$\vec{E}_H = -(\Omega_F - \Omega_H)\,\tilde{\omega}\,\vec{e}_\varphi \times \vec{B}_\perp. \tag{205}$$

The representations (197) of $\mathcal{B}$, and (204) for $\mathcal{E}$ will be important in the final section 9.

Now, we proceed as earlier in section 5 and consider again the closed path $\mathcal{C}$ in Fig. 11.

We already had the relations

$$\triangle V = \frac{1}{2\,\pi}\,\Omega_H\,\triangle\Psi, \tag{206}$$

and

$$\triangle R_H = R_H\,\frac{\triangle\Psi}{4\,\pi^2\,\tilde{\omega}^2\,B_\perp} \tag{207}$$

(see (127) and (130)). As in (143) we obtain for the horizon voltage

$$\triangle V_H = I\,\triangle R_H. \tag{208}$$

This contribution can alternatively be computed with (204):

$$\triangle V_H = \int_{\mathcal{C}_H} \alpha\,\mathcal{E} = \frac{\Omega_H - \Omega_F}{2\,\pi}\,\triangle\Psi. \tag{209}$$

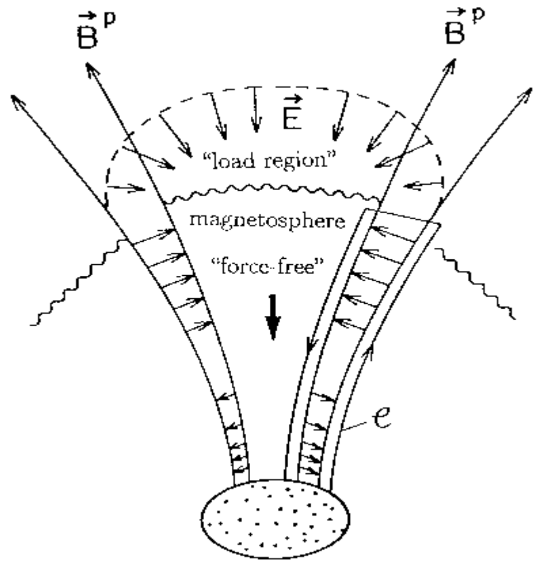

Figure 11: Magnetosphere of a rotating BH (adapted from [1]).

The total voltage $\triangle V$ of $\mathcal{C}$ is the sum of $\triangle V_H$ and the voltage drop $\triangle V_L$ in the astrophysical load region (see Fig. 11). For the latter we obtain from (204) (since $\omega \simeq 0$)

$$\triangle V_L = \frac{1}{2\pi}\,\Omega_F\,\triangle\Psi \tag{210}$$

and, of course, also

$$\triangle V_L = I\,\triangle R_L, \tag{211}$$

where $\triangle R_L$ is the resistance of the load region. From this equations, and

$$\triangle V = \triangle V_H + \triangle V_L, \tag{212}$$

we immediately find the relations

$$\frac{\triangle V_L}{\triangle V_H} = \frac{\Omega_F}{\Omega_H - \Omega_F} = \frac{\triangle R_L}{\triangle R_H}, \tag{213}$$

$$I = \frac{\triangle V}{\triangle R_H + \triangle R_L} = \frac{1}{2}\,(\Omega_H - \Omega_F)\,\tilde{\omega}^2\,B_\perp. \tag{214}$$

The ohmic dissipation at the horizon is as in (179)

$$\begin{aligned} T_H\,\frac{d\,(\triangle S_H)}{dt} &= I^2\,\triangle R_H = I\,\triangle V_H \\ &= \frac{(\Omega_H - \Omega_F)^2}{4\,\pi}\,\tilde{\omega}^2\,B_\perp\,\triangle\Psi. \end{aligned} \tag{215}$$

The power $\triangle P_L$ deposited in the load is thus

$$\triangle P_L = I^2\,\triangle R_L = I\,\triangle V_L \stackrel{(210),(214)}{=} \frac{\Omega_F\,(\Omega_H - \Omega_F)}{4\,\pi}\,\tilde{\omega}^2\,B_\perp\,\triangle\Psi. \tag{216}$$

As in (153), $\triangle P_L$ becomes maximal for $\triangle R_H = \triangle R_L$, which is the case for (see (213))

$$\Omega_F = \frac{1}{2}\,\Omega_H. \tag{217}$$

Then the estimates (133) and (135) hold.

Whether the crucial condition (217) is approximately satisfied in realistic astrophysical scenarios is a difficult problem for model builders. Therefore, the question remains open whether the process proposed by Blandford and Znajek [9] is important for the energy production in active galactic nuclei. It could, however, play a significant role in the formation of relativistic jets.

In this connection it should be mentioned that the Blandford-Znajek process may play an important role in gamma-ray bursts, as has been suggested, for instance by Mészáros and Rees [10]. One of the proposed models involves the toroidal debris from a disrupted neutron star orbiting around a BH. This debris may contain a strong magnetic field, perhaps amplified by differential rotation, and an axial magnetically-dominated wind may be generated along the rotation axis which would contain little baryon contamination. Energized by the BH via the Blandford-Znajek process, a narrow channel Poynting-dominated outflow with little baryon loading could be formed. The latter property is crucial for explaining the efficient radiation as gamma rays. (For a review of this topic, see for instance [11].)

9 Axisymmetric, Stationary Electrodynamics of Force-Free Fields

In this concluding section we discuss the electrodynamics of force-free fields around a BH a bit more systematically. The main goal is to derive the general relativistic *Grad-Shafranov equation*, whose nonrelativistic limit plays an important role in the electrodynamics of pulsars.

For *stationary* fields, Maxwell's equations (38) become

$$\mathbf{d}\mathcal{B} = 0, \qquad \mathbf{d}(\alpha\,\mathcal{E}) = L_\beta\,\mathcal{B}, \tag{218}$$

$$\mathbf{d} * \mathcal{E} = 4\pi\rho, \qquad \mathbf{d}(\alpha * \mathcal{B}) = 4\,\pi\,\alpha\,\mathcal{J} - L_\beta * \mathcal{E}. \tag{219}$$

¿From now on we assume also *axisymmetry*. Then $L_\beta\,\rho = \mathbf{d}\,i_\beta\,\rho = 0$, and hence the continuity equation (24) reduces to

$$\mathbf{d}\,(\alpha\,\mathcal{J}) = 0. \tag{220}$$

I recall the representation (197) of $\mathcal{B}$ in terms of the potential Ψ and I:

$$\mathcal{B} = \frac{1}{2\,\pi}\,\mathbf{d}\Psi \wedge \mathbf{d}\varphi + \frac{2\,I}{\alpha} * \mathbf{d}\varphi. \tag{221}$$

Ψ and I are independent of φ; their physical significance has already been discussed: Using the notation in Fig. 10 we have

$$\Psi = \int_{\mathcal{A}} \mathcal{B}, \qquad I = \int_{\mathcal{A}} \alpha\,\mathcal{J}. \tag{222}$$

At this point we make the simplifying assumption that the ideal MHD condition (193) holds *everywhere*. The fields are thus force-free, and we know from section 8 that the $\mathcal{E}$-field is poloidal and can be represented as (see (204))

$$\alpha\,\mathcal{E} = -\frac{\Omega_F - \omega}{2\,\pi}\,\mathbf{d}\Psi, \tag{223}$$

The representation (200) of the poloidal current

$$\alpha\,\mathcal{J}^{\mathrm{pol}} = \frac{1}{2\,\pi}\,\mathbf{d}I \wedge \mathbf{d}\varphi. \tag{224}$$

was obtained without assuming the ideal MFD condition. But if this is assumed, the toroidal part of $\vec{j} \times \vec{B}$ in (193) has to vanish, which means that $\mathcal{J}^{\mathrm{pol}}$ and $\mathcal{B}^{\mathrm{pol}}$ are proportional to each other. Comparison of (221) and (224) shows that $\mathbf{d}I$ must then be proportional to $\mathbf{d}\Psi$, thus I is a function of Ψ alone:

$$I = I(\Psi). \tag{225}$$

We have shown already in section 8 that Ω_F is also a function of Ψ,

$$\Omega_F = \Omega_F(\Psi). \tag{226}$$

According to (221) and (224) we now have

$$\mathcal{J}^{\mathrm{pol}} = \frac{1}{\alpha}\,\frac{dI}{d\Psi}\,\mathcal{B}^{\mathrm{pol}}. \tag{227}$$

The total current $\vec{j}$ can now be represented as

$$\vec{j} = \rho_e\,\vec{v}_F + \frac{1}{\alpha}\,\frac{dI}{d\Psi}\,\vec{B}. \tag{228}$$

In view of (227), the poloidal part of this equation is certainly correct, and the toroidal part on the right hand side is chosen such that $\vec{j} \times \vec{B} = \rho_e\,\vec{v}_F \times \vec{B} = -\rho_e\,\vec{E}$ (see (201)), which is just the ideal MFD condition.

Our goal is now to derive – for given functions (225) and (226) – a partial differential equation for the potential Ψ. We shall achieve this by computing the toroidal part of the current in two independent ways. First, we take the toroidal part of (228) and obtain with $B^{\mathrm{tor}} = \frac{2\,I}{\alpha\,\tilde{\omega}}$ (see (221))

$$j^{\mathrm{tor}} = \rho_e\,\frac{\Omega_F - \omega}{\alpha}\,\tilde{\omega} + \frac{1}{\alpha}\,\frac{dI}{d\Psi}\,\frac{2\,I}{\alpha\,\tilde{\omega}}.$$

Here, we also eliminate ρ_e: From (223) and (219) we deduce

$$8\pi^2\rho = -\mathbf{d} * \left(\frac{\Omega_F - \omega}{\alpha}\,\mathbf{d}\Psi\right), \tag{229}$$

i.e.,

$$8\pi^2\rho_e = -\vec{\nabla}\cdot\left(\frac{\Omega_F - \omega}{\alpha}\,\vec{\nabla}\Psi\right). \tag{230}$$

We thus arrive at

$$j^{\rm tor} = -\frac{1}{8\pi^2}\,\frac{\Omega_F-\omega}{\alpha}\,\tilde{\omega}\,\vec{\nabla}\cdot\left(\frac{\Omega_F-\omega}{\alpha}\,\vec{\nabla}\Psi\right) + \frac{2\,I}{\alpha^2\,\tilde{\omega}}\,\frac{dI}{d\Psi}. \tag{231}$$

On the other hand, this quantity can also be obtained from Maxwell's equation (219):

$$4\pi\alpha\,\mathcal{J}^{\rm tor} = [\mathbf{d}(\alpha * \mathcal{B})]^{\rm tor} + [L_\beta * \mathcal{E}]^{\rm tor}. \tag{232}$$

For the first term on the right we have

$$2\pi\,[\mathbf{d}(\alpha * \mathcal{B})]^{\rm tor} = 2\,\pi\,\mathbf{d}(\alpha * \mathcal{B}^{\rm tor}) \overset{(221)}{=} \mathbf{d}\,[\alpha * (\mathbf{d}\Psi \wedge \mathbf{d}\varphi)]. \tag{233}$$

Now, we use the following useful general identity, whose proof is left as an exercise: Let χ be a p-form and $\vec{m}$ a Killing field. Assume also $L_{\vec{m}}\,\chi = 0$, then

$$\delta\,(m \wedge \chi) = -m \wedge \delta\,\chi, \tag{234}$$

where δ is the codifferential.

For $\chi = (\alpha/\tilde{\omega}^2)\,\mathbf{d}\Psi$, $\vec{m} = \vec{\partial}_\varphi$, $m = \tilde{\omega}^2\mathbf{d}\varphi$ this gives

$$\delta\,(\alpha\,\mathbf{d}\Psi \wedge \mathbf{d}\varphi) = \delta\,\left(\frac{\alpha}{\tilde{\omega}^2}\,\mathbf{d}\Psi\right)\,\tilde{\omega}^2\,\mathbf{d}\varphi$$

or

$$\mathbf{d} * [\alpha\,\mathbf{d}\Psi \wedge \mathbf{d}\varphi] = \delta\left(\frac{\alpha}{\tilde{\omega}^2}\,\mathbf{d}\Psi\right)\tilde{\omega}^2 * \mathbf{d}\varphi.$$

Thus, (233) becomes

$$2\pi\,\mathbf{d}(\alpha * \mathcal{B}^{\rm pol}) = -\vec{\nabla}\cdot\left(\frac{\alpha}{\tilde{\omega}^2}\,\vec{\nabla}\Psi\right)\,\tilde{\omega}^2 * \mathbf{d}\varphi. \tag{235}$$

Next, we turn to the second term in (232). By (223) we have

$$-L_\beta * \mathcal{E} = L_\beta\,\left[\frac{\Omega_F-\omega}{2\pi\alpha} * \mathbf{d}\Psi\right].$$

If χ denotes the square bracket, we can write $L_\beta\,\chi = L_{-\omega\,\partial_\varphi}\,\chi = -\omega\,L_{\partial_\varphi}\,\chi - \mathbf{d}\omega \wedge i_{\partial_\varphi}\,\chi$, and obtain

$$*L_\beta * \mathcal{E} = -\frac{\Omega_F-\omega}{2\,\pi\,\alpha}\,i_{\vec{\nabla}\,\omega}\,(\mathbf{d}\Psi \wedge \tilde{\omega}^2\,\mathbf{d}\varphi). \tag{236}$$

With (235) and (236) we obtain for the Hodge-dual of (232)

$$4\pi\alpha * \mathcal{J}^{\rm tor} = -\frac{1}{2\,\pi}\,\vec{\nabla}\cdot\left(\frac{\alpha}{\tilde{\omega}^2}\,\vec{\nabla}\Psi\right)\,\tilde{\omega}^2\,\mathbf{d}\varphi - \frac{\Omega_F-\omega}{2\,\pi\,\alpha}\,\left(\vec{\nabla}\,\omega\cdot\vec{\nabla}\,\Psi\right)\,\tilde{\omega}^2\,\mathbf{d}\varphi.$$

Since $*\mathcal{J}^{\rm tor} = j^{\rm tor}\,\frac{1}{\tilde{\omega}}\,\tilde{\omega}^2\,\mathbf{d}\varphi$, we get

$$\frac{4\pi\alpha}{\tilde{\omega}}\,j^{\rm tor} = -\frac{1}{2\,\pi}\,\vec{\nabla}\cdot\left(\frac{\alpha}{\tilde{\omega}^2}\,\vec{\nabla}\Psi\right) - \frac{\Omega_F-\omega}{2\,\pi\,\alpha}\,\left(\vec{\nabla}\,\omega\cdot\vec{\nabla}\,\Psi\right). \tag{237}$$

Inserting $\vec{\nabla}\omega = \vec{\nabla}(\omega - \Omega_F) + (d\Omega_F/d\Psi)\,\vec{\nabla}\Psi$ leads finally to our second formula for $j^{\rm tor}$:

$$\begin{aligned} 8\pi^2 j^{\rm tor} &= -\frac{\tilde{\omega}}{\alpha}\,\vec{\nabla}\cdot\left(\frac{\alpha}{\tilde{\omega}^2}\,\vec{\nabla}\Psi\right) + \frac{\tilde{\omega}}{\alpha^2}\,(\Omega_F - \omega)\,\vec{\nabla}\Psi\cdot\vec{\nabla}(\Omega_F - \omega) \\ &\quad -\frac{\tilde{\omega}}{\alpha^2}\,(\Omega_F - \omega)\,\frac{d\Omega_F}{d\Psi}\,(\vec{\nabla}\Psi)^2. \end{aligned} \tag{238}$$

Comparison of (232) with (238) gives, after a few steps, the following *generalized Grad-Shafranov equation*:

$$\vec{\nabla}\cdot\left\{\frac{\alpha}{\tilde{\omega}^2}\left[1 - \frac{(\Omega_F - \omega)^2\,\tilde{\omega}^2}{\alpha^2}\right]\vec{\nabla}\Psi\right\} + \frac{\Omega_F - \omega}{\alpha}\,\frac{d\Omega_F}{d\Psi}\,(\vec{\nabla}\Psi)^2 + \frac{16\,\pi^2}{\alpha\,\tilde{\omega}^2}\,I\,\frac{dI}{d\Psi} = 0. \tag{239}$$

The integration of the original equations (for axisymmetric stationary situations) is reduced to this single partial differential equation. $I(\Psi)$ and $\Omega_F(\Psi)$ are free functions[2], and if Ψ is a solution of (239) the electromagnetic fields are given by (221) and (223), while the charge and current distribution can be obtained from (230), (224), and (231).

A limiting case of (239) is known from the electrodynamics of pulsars: Ignoring the curvature of spacetime and using that Ω_F is equal to the angular velocity Ω of the neutron star (to derived shortly), we get the *pulsar equation*:

$$\vec{\nabla}\cdot\left\{\frac{1}{\tilde{\omega}^2}\left[1 - \Omega^2\,\tilde{\omega}^2\right]\vec{\nabla}\Psi\right\} + \frac{16\,\pi^2}{\tilde{\omega}^2}\,I\,\frac{dI}{d\Psi} = 0 \tag{240}$$

($\tilde{\omega}$ is the radial cylindrical coordinate in flat space).

The equation (239) holds also outside an aligned pulsar, since all the basic equations in section 2 remain valid there. However, the different boundary condition at the surface of the neutron star implies $\Omega_F = \Omega$, as we now show. In the interior of the neutron star the 3-velocity is (see (203))

$$\vec{v} = \frac{1}{\alpha}\,(\Omega - \omega)\,\vec{\partial}_\varphi. \tag{241}$$

Since the neutron star matter is ideally conducting, the electric field there is

$$\mathcal{E} = i_{\vec{v}}\,\mathcal{B} = -\frac{1}{2\pi\alpha}\,(\Omega - \omega)\,\mathbf{d}\Psi,$$

if the $\mathcal{B}$-field is poloidal (first term in (221)). At the boundary this has to agree with (223), implying $\Omega_F = \Omega$.

Another remark should be made at this point about the interior of the neutron star. We showed in section 8 that $\mathbf{d}\Omega_F \wedge \mathbf{d}\Psi = 0$ implying $\mathbf{d}\Omega \wedge \mathbf{d}\Psi = 0$ inside the star. It will, in general, not be possible to represent Ψ as a function of Ω, since Ψ has to satisfy the Grad-Shafranov equation for $I(\Psi) = 0$ inside the star. Therefore, the rotation must be *rigid*, $\Omega = \text{const}$.

[2]These are, of course, restricted by boundary conditions, but we do not discuss this here.

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